CASIO.

Limits – Calculation and Visualization

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LEVEL
High school or university students starting with Calculus.

OBJECTIVES
To comprehend the notion of a limit by drawing graphs and performing their calculations.

Corresponding eActivity
C02ACT1.g1e (for Activity 1), C02ACT2.g1e (for Activity 2), C02ACT3.g1e (for Activity 3)

OVERVIEW
Intuitively, limits are results of complex stepwise process that factually never reaches the target point. For students, it is rather difficult to grasp the relationship between the approximation process and its resulting value. To facilitate their comprehension, same calculations are performed using two different methods: a "long calculation" (simulating the stepwise approximation) and by a formal (algebra-based) method. The problems demonstrate the limit of a function at infinity as well as the limit of a function in a given point.

[Note]
1. We shall use small letter x instead of capital X as shown on the calculator throughout the paper.
2. The problems from [Tan] have been reformulated and solved using different methods.

EXPLORATORY ACTIVITIES

Activity 1 (C02ACT1.g1e):
The Custom Office Company makes a line of executive desks. The fixed cost of production is $ 200,000 per year, the costs of material and workforce for one desk is $ 100. Show that:
a) With the growing production, the unit production cost per year declines.
b) Show that there is a lower bound of this cost.
c) Interpret the result.

Solution:

The total cost of production of x desks during one year is expressed by the formula

$$C(x) = 100x + 200000$$

The cost per piece is then expressed by the function

$$A(x) = 100 + \frac{200000}{x}$$

We use the limit notation to express the dependence between the cost per piece and the growing number of pieces

$$\lim_{x \to \infty} A(x) = \lim_{x \to \infty} \left(100 + \frac{200000}{x} \right)$$

The limit notation indicates that the upper bound of production does not exist and we want to calculate its "farthest" value.

A. Estimating the range of functional values using a spreadsheet calculation:

(a) (Refer to Table 1) Let us open the spreadsheet tool and fill in the first few values – small numbers like 1, 2, 3 and 4 in the A column – and calculate their corresponding functional values in the column B:

With growing number of pieces, the functional values are becoming smaller, but for such small series they are apparently too large e.g. producing 3 pieces per year would result in the cost pre piece equal to $ 66 766. To make the production effective, the series must be much bigger. In e-Activity, Table 1 shows the values for all *x* up to 100.

(b) (Refer to Table 2) To make the size of the series flexible and to become capable of going to even larger numbers, let us reorganize the working area of the spreadsheet. The cell A1 will contain the beginning of the interval, the column B the production per year and the C column the price per piece.
In the below example, A1 contains 1000. The cell B1 refers directly to it (using the formula =A1), the remaining cells in the column B refer to their upper neighbors (starting with =B1+1 in B2). The functional values are calculated in the same way as before (with the cells referring to their left neighbors).

The changes of the value in A1 are automatically reflected in the column C. For example, replacing 1000 by 10000 results in the next values:

The production cost values are smaller with each additional piece but the differences between two consecutive ones become almost negligible. Still, the big moves in A1 will result in observable changes in them. (Try 100,000, 1,000,000, etc.)

(c) (Refer to <u>Growing by 10%</u>) Even higher flexibility can be achieved by introducing a pair of parameters – the starting point and an increment. To simplify our task, we use the increment equal 10% of the starting point value.
Thus, the value of A1 will be an arbitrary (but positive) input value. All the values in the B column will be calculated by using the 10% increment. B1 will be evaluated by using the formula = A1; B2 using the formula =B1+0.1*A1. The latter formula is copied to the below cells.

(d) (Refer to <u>Growing by 10%</u>) Change the value in A1 to even bigger numbers and watch what happens in the column C. Try to generalize your observations.

EXERCISES A
Exercise 1.
In the "Growing by 10%" activity, set big values (10,000, 100,000, 1,000,000 etc.) into A1. The functional values become very close to 100. Can you produce values smaller than 100?

Exercise 2.
Formulate a hypothesis on behavior of the function for very big values of *x*. Start with formulating your answer to the question: Can 97 become a value in the C column? Why not?

SOLUTIONS to EXERCISES A
Exercise 1.
No, it is impossible. See Exercise 2 for details.

Exercise 2.
No, it is always bigger than 100. The function formula is 100 + (200 000/x). As x is positive, the second summand is positive, too. So the result is always bigger than 100. Naturally, for very big x only negligibly.

B. Graphing the function:

(a) (Refer to the Viewing it) In the previous section we have learned that:
- Only the positive half of x axis is to be displayed.
- The displayed interval should be very long (up to tens of thousand) to give a realistic view of the function behavior.
- The range of functional values on this interval goes from 200,000 to 100.

Open the graph editor and graph the function *A(x)*. In View Window, we set up appropriate parameters. The set used below is:
-   x varies between 1,000 and 10,000.
-   y varies from 0 to 300.
Then we obtain the following graph:
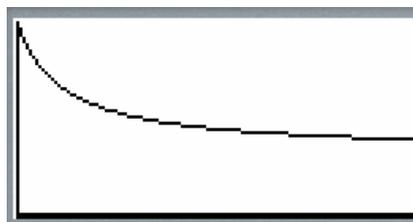
The graph nicely demonstrates that the slope of the curve declines slower and slower.

EXERCISES A
Exercise 3.
Set up the values of X_{min} and X_{max} to different (even higher) values covering large intervals (at least several thousand units). Do not change Y_{min} and Y_{max}. Notice that with growth of *x* the slope becomes gentler and gentler

SOLUTIONS to EXERCISES A
Exercise 3.
The curve is much closer to a straight line because all functional values are now close to 100.

C. Analytical Approach:
(a) Calculate the limit manually:

$$\lim_{x \to \infty} \left(100 + \frac{200000}{x} \right) =$$

(b) Discuss the outcome: The limit will never be reached. So, why are we interested in the result?

Activity 2 (C02ACT2.g1e):
Due to a danger of dividing by 0, the value of the function

$$\frac{(x-4)(x+1)}{\sqrt{x} - 2}$$

is not defined at the point $x = 4$. What is really happening in its neighborhood? Can we estimate (or even calculate) its "appropriate" value?

Solution:
If such a value exists, it is expressed as the limit

$$\lim_{x \to 4} \frac{(x-4)(x+1)}{\sqrt{x} - 2}$$

The subject of our discussion is the understanding what the notation determines.

A. Watching behavior of the function in a spreadsheet table:

(a) (Refer to TABLE 1) TABLE 1 contains the functional values from a neighbourhood near $x = 4$. For obvious reasons, the spreadsheet field corresponding to f(4) is empty – typing the formula into it would cause an error (an attempt to divide by zero). Notice that the picture is artificially enlarged using a graphic editor. At the calculator screen no more than 5 rows are displayed.

SHEE	A	B	C	D
1	3.5	17.418		
2	3.6	17.927		
3	3.7	18.44		
4	3.8	18.956		
5	3.9	19.476		
6	4			
7	4.1	20.526		
8	4.2	21.056		
9	4.3	21.59		
10	4.4	22.127		

=(A1−4)×(A1+1)÷(√A1−2)

(b) The B6 cell is intentionally empty. The FX-9860 SD calculator does not allow drawing

a graph from the sheet with a missing field. So, we have to assign a value into B6. As one easily see from the above picture, a simple comparison of its predecessors (B1 to B5) and successors (B7 to B10) indicates the value of 20 as a good candidate.

(c) (Refer to Graph using 20) To draw a line one has to set up its parameters (using SET) to the form in the left. After return to the worksheet and pressing GPH1 the figure on the right is displayed.
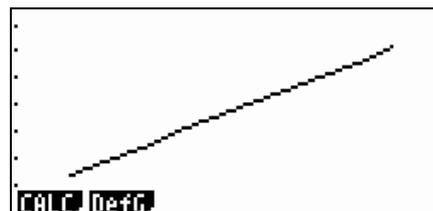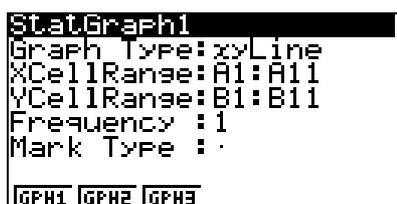
Evidently, using the value "20" completes the function in a smooth manner.

B. Approaching the critical point from even closer distances:

(a) (Refer to Modify A1) We open the spreadsheet the next time. Its data are now arranged in a different pattern. It is capable of displaying points from neighborhoods specified by the user.

The investigated point is placed in B5 – in the middle between its neighbors. Its predecessors are in cells B1 to B4; its ancestors are in cells B6 to B6.
The A1 cell contains an equidistant step between each pair of consecutive values.
The column C contains their functional values calculated (with an exception for $x=4$ which is artificially added as before). For example, the formula in C1 is
=(B1 − 4)*(B1 + 1)/(√B1 − 2)

See the next picture to observe the difference.

SHEE	A	B	C	D
1	0.01	3.96	19.79	
2		3.97	19.842	
3		3.98	19.895	
4		3.99	19.947	
5		4	20	
6		4.01	20.052	
7		4.02	20.105	
8		4.03	20.157	
9		4.04	20.21	

(b) Let us experiment with the content of A1. Change its value to 0.1, 0.01, 0.001, etc. The values of the column B and C are automatically changing, too. Still, something does not change:
- In every selected neighborhood of 4, all functional values in C1 to C4 are smaller than 20, whilst the functional values in B6 to B9 are all bigger than 20.
- The smaller the neighborhood, the closer are its functional values to 20.

Informally speaking, the value "20" is the edge dividing the functional values smaller than 20 (on the left side, for x < 4) from the functional values greater than 20 (on the right side, for x > 4). So, despite the fact the function is undefined for x = 4, its value can be expressed as a limit

$$\lim_{x \to 4} \frac{(x-4)(x+1)}{\sqrt{x}-2} = 20$$

C. Analytical Approach:

Calculate the value of

$$\lim_{x \to 4} \frac{(x-4)(x+1)}{\sqrt{x}-2} = \lim_{x \to 4} \left((\sqrt{x}+2)(x+1) \right) = 4.5 = 20$$

EXERCISE B
Exercise 1.
Does the function

$$f(x) = \frac{x^2 - 9}{x + 3}$$

have a limit at x=−3? If so, what is its value?

SOLUTION to EXERCISE B
Exercise 1.
Using spreadsheet approximations, it is easy to show that the limit is -6.

Activity 3 (C02ACT3.g1e):

The absent value resulted from a division by 0, cannot always be replaced by a particular value. So the limit not always exists.
Let us discuss another example:

$$\lim_{x \to -3} \frac{x+1}{x+3}$$

The function is not defined at the point *x* = −3. What is happening in its neighborhood?

Solution:

A. Watching behavior of the function in a spreadsheet table:

(a) (Refer to Values near -3) Similarly to Activity 2, the behavior of the function in the neighborhood of −3 can be investigated using a spreadsheet. For same reasons, we left

the spreadsheet field corresponding to f(−3) empty and then try to estimate the value that could be entered to it.

SHEE	A	B	C	D
1	-3.5	5		
2	-3.4	6		
3	-3.3	7.6666		
4	-3.2	11		
5	-3.1	21		
6	-3			
7	-2.9	-19		
8	-2.8	-9		
9	-2.7	-5.666		
10	-2.6	-4		

=(A1+1)÷(A1+3)

Compared to Activity 2, the column B shows a different picture. The closer we go to the point x = −3, the bigger differences between the values on the left and right sides of it appear. The left-side values grow from 5 to 21, whilst the right-side values change from -4 to -19 as we approach -3.

(b) Does the picture change in very close neighborhoods? Again, allow the flexible borders of the neighborhood and watch the changes.

(Refer to closer to -3) Again, let us experiment with the content of A1. Change its value to 0.1, 0.01, 0.001, etc. The values of the column B and C are automatically changing, too. Now the invariants of the activity differ:
- The left neighbors of −3 are always positive, its right neighbors are always negative. In theory, the value 0 could be a candidate for the limit. However, we observe the followings:
- The closer we go to −3, the bigger are differences between the nearest neighbors. In the above picture, the value of the left one is 21, the value of the right one is -19. The next picture shows that the difference between two closer neighbors is even bigger.

SHEE	A	B	C	D
1	0.01	-3.04	51	
2		-3.03	67.666	
3		-3.02	101	
4		-3.01	201	
5		-3		
6		-2.99	-199	
7		-2.98	-99	
8		-2.97	-65.66	
9		-2.96	-49	

So, we conclude that the limit value of the left neighborhood is +∞, but the limit value of the right neighborhood is −∞. Thus, there is no value that is approached from both sides. In such a case we say that the limit does not exist.

EXERCISES C
Exercise 1.
Show that the function

$$f(x) = \frac{(x+17)}{x^2 - 1}$$

does not have a limit at x=1.

Exercise 2.
The limit only exists when it can be fixed as a constant. Sow that the function

$$g(x) = \frac{1}{x^2}$$

does not have a limit at x=0.

SOLUTIONS to EXERCISES C
Exercise 1.
Using a spreadsheet, investigate the function at a small neighborhood of 1. Show that the values do not progress to a constant

Exercise 2.
Investigations show that all values in the neighborhoods of 0 are positive but they are bigger and bigger. There is no constant that would limit (stop) the growth. So there is no limit.

REFERENCE
[Tan] S. T. Tan, *College Mathematics* (Chapter 9), PWS-KENT Publishing Company, 1998. ISBN 0-534-91791-7.