Applications to Integration

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OBJECTIVES
In this paper, we study applications of the definite integral with the aid of the graphics calculator.

Corresponding eActivity
area.g1e

OVERVIEW
One of the applications of definite integration is the notion of the average value of a given function. If the function f is integrable on the closed interval $[a,b]$ then the **average value V of f** on $[a,b]$ is defined as

$$V = \frac{\int_a^b f(x)dx}{b-a}$$

In this paper, we study concepts pertaining to the average value of a given function from the numerical and geometric points of view with the aid of the graphics calculator.

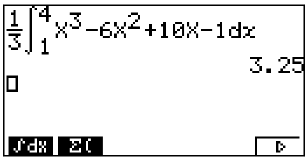
EXPLORATORY ACTIVITIES
Example. Let $f(x) = x^3-6x^2+10x-1$.

1. Find the average value of the function f on the interval [1,4].
2. Determine the value(s) of x at which the average value occurs.
3. Describe the geometric interpretation of the results.

Solution:
1. We substitute $f(x) = x^3-6x^2+10x-1$, a = 1 and b = 4 directly in the formula for the average value V given earlier:

$$V = \frac{\int_a^b f(x)dx}{b-a}$$

In the Run Editor, we access [Math $\int$ dx] and enter the expression as follows:



The answer we obtain is 3.25. The average value of *f* at [1,4] is 3.25.

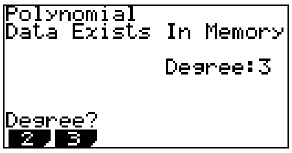
2. The value(s) of x at which the average value occurs may be obtained by solving the equation

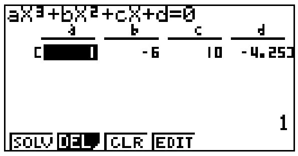
$$x^3\text{-}6x^2\text{+}10x\text{-}1 = 3.25$$

or equivalently,

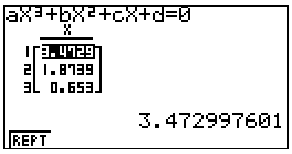
$$x^3\text{-}6x^2\text{+}10x\text{-}4.25 = 0$$

Using the Solver (open **First solution**->press Exit key and press F2) we enter the third degree polynomial in this manner:



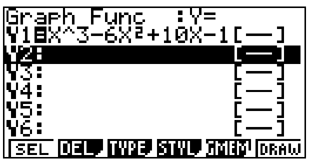


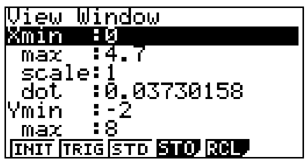
We press [Solv] to obtain the solutions to the cubic equation. We obtain the approximate values of x as follows: 3.47, 1.8739 and 0.653.



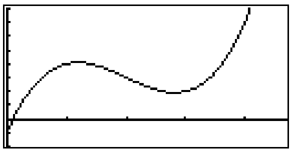
However, since our given interval is [1,4], we only consider the values $x \approx 1.8739$, 3.4729.

Another way to obtain these answers is to graph the function $y = x^3\text{-}6x^2\text{+}10x\text{-}1$ in the Graph Editor (open **2nd solution**) using an appropriate View Window:

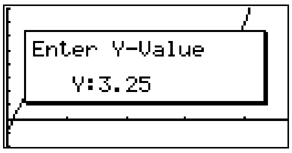


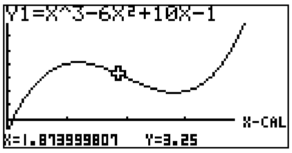


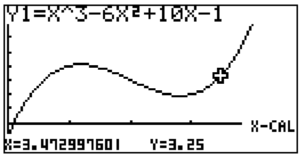
The graph is given as follows:



Using [SHIFT GSolv XCal], we can solve for the x values corresponding to y = 3.25 as follows:



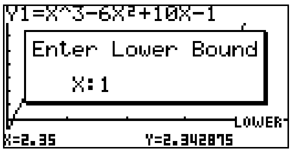


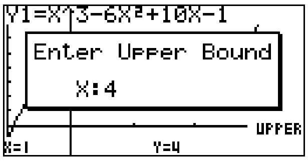


The answers we get are the same values obtained earlier, $x \approx$ 1.8739, $x \approx$ 3. 4729.

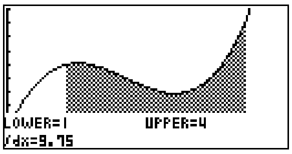
3. Geometric Interpretation of Average Value:

The definite integral $\int_1^4 f(x)dx$, found as an area between the curve of $f(x)$ and the x axis between the x values 1 and 4, can be obtained using the command [SHIFT G-Solv $\int$ dx]. The lower and upper bounds of integration are entered after pressing [2].

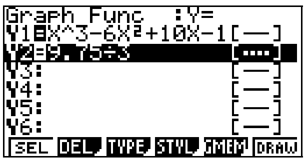


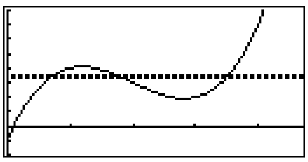


The value of the definite integral $\int_1^4 f(x)dx$ is given by 9.75 square units.

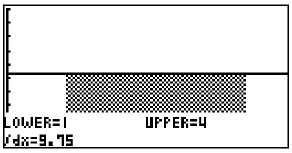


Note that if we graph $\dfrac{\int_1^4 f(x)dx}{b-a} = \dfrac{9.75}{3}$ =3.25, we obtain a horizontal line.

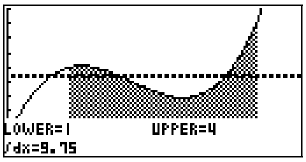


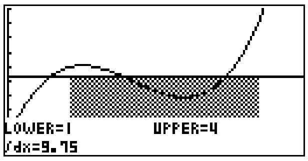


Finding the area between this horizontal line and the *x* axis between the *x* values *x* =1 and *x* =4, is the same as getting the area of the rectangle having height 3.25 and width 3.



Observe that the area of this rectangle is equal to the area represented by the definite integral $\int_1^4 f(x)dx$ calculated earlier.

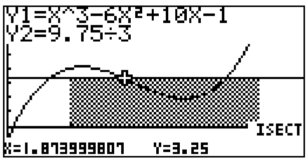


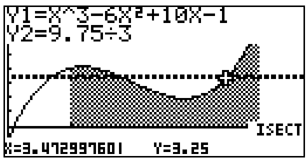


Now, from the formula of average value *V*, we have $(b-a)V = \int_1^4 f(x)dx$. This implies that the area of the rectangle is (*b*-*a*)*V* or 3*V*.

Thus, the average value *V* represents the height of the rectangle whose base is the interval $1 \le x \le 4$ and whose area is the same as the area under the curve *f*(*x*) over $1 \le x \le 4$.

The average value of *f* occurs at *x* ≈ 1.873 and *x* ≈ 3.4729.





These points are obtained by getting the points of intersection of both curves, [SHIFT GSolv ISCT].

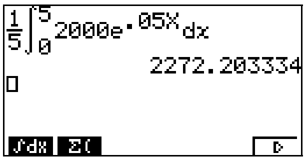
EXERCISES:

1.The number of bacteria present in a certain culture after *x* minutes of an experiment was *Q*(*x*) = 2,000$e^{0.05x}$. What was the average number of bacteria present during the first 5 minutes of the experiment? At what time into the first 5 minutes of the experiment does the average number of bacteria occurs?
2. Use integrals to approximate a rectangle whose area is that of a semi-circle of radius 7.

SOLUTIONS:

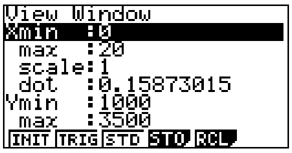
Exercise 1.

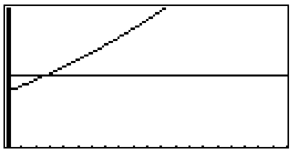
We calculate the average value in the Run Editor as follows:



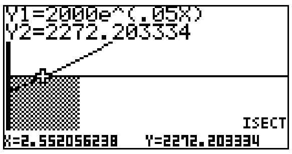
The average number of bacteria in the first 5 minutes is approximately 2,272.203334.

We graph the functions $y = 2{,}000e^{0.05x}$ and $y = \dfrac{\int_0^5 2000e^{0.05x}dx}{5}$ =2272.203334 in the Graph Editor as follows using an appropriate Graph Window:

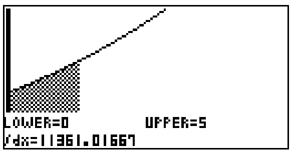


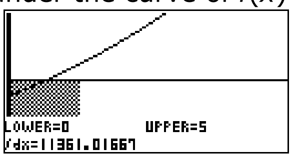


By getting the intersection of both curves, we observe that during approximately 2.55 minutes into the experiment, the number of bacteria attains its average value which is, approximately 2272.203334:



Geometrically, the average value is the height of the rectangle whose base is the interval $0 \le x \le 5$ and whose area is the same as the area under the curve of $f(x)$ over $0 \le x \le 5$

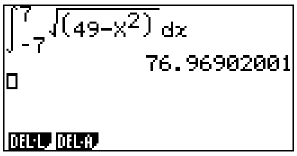


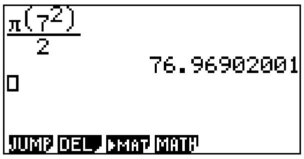


Exercise 2.

The value of the definite integral $\int_{-7}^{7}\sqrt{49-x^2}\,dx$ approximates the area of the semicircle of radius 7 which we situate having center at the origin. From geometry, we recall that the area is also given by the geometric formula $\frac{\pi \cdot 7^2}{2}$.

In the Run Editor, we calculate this area to be equal to approximately 76.96902001.





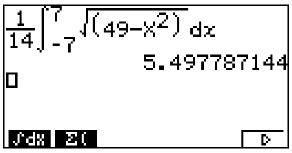
Using the formula for the average value *V* of the function $\sqrt{49-x^2}$ on the interval [-7,7], we have the expression

$$(7-(-7))V = \int_{-7}^{7}\sqrt{49-x^2}\,dx$$

or equivalently,

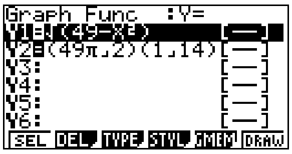
$$49V = \int_{-7}^{7}\sqrt{49-x^2}\,dx$$

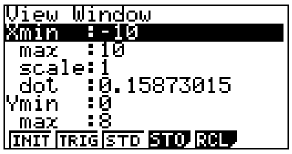
In the Run Editor, we find that $V = \frac{\int_{-7}^{7}\sqrt{49-x^2}}{14} \approx 5.4977$



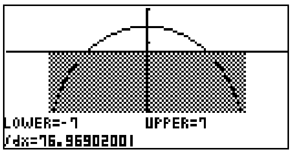
The rectangle we want has width 14 and height approximately 5.4977.

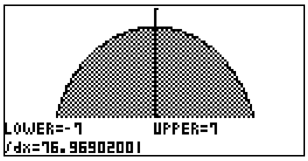
In the Graph Editor, we can visualize the situation by graphing $y = \sqrt{49-x^2}$ and $y = \frac{\pi \cdot 7^2}{2}\frac{1}{14}$,(we use this value to represent the height), and obtaining the areas under the curve.



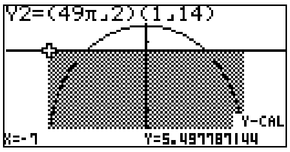


Observe that the areas of the rectangle and semicircle are the same:





Note that the height of the rectangle is approximately 5.4977.



REFERENCE

[1] Hoffmann, Laurence, et al. *Calculus*, 8th Edition. McGraw Hill, 2004.

