Hyperbolic Functions

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LEVEL
High school or university students with basic knowledge on functions and conic sections.

OBJECTIVES
To explain the reasons for introducing complex functions.

Corresponding eActivity
F10ACOSH.g1e (for Activity 1), F10ARCH.g1e (for Activity 2).

OVERVIEW
Basic properties of hyperbolic functions are demonstrated. Their importance for practical applications is accented. It is shown that despite their rather complex expression by exponential functions, they model situations from the world around us.

EXPLORATORY ACTIVITIES
[Note]
For variables, we shall use small letters *x* and y instead of capital X as shown on the calculator throughout the paper.

The stress in eActivities is done on visualization of hyperbolic functions and on the fact that they are examples of discoveries of great mathematicians derived from problems erroneously solved by other great mathematicians.

Activity 1 (F10ACOSH.g1e):

EXERCISES A

***Exercise 1*.**

Take a piece of rope and hold its ends in your hands. Move your hands closer and further from each other. Look at the rope and think about its shape. Which curve could it be?

SOLUTIONS to EXERCISES A

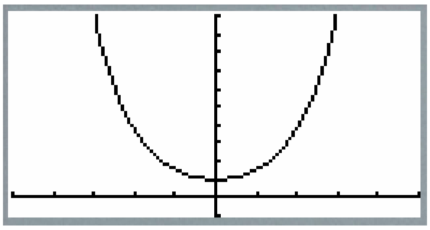
***Exercise 1*.**

The curve looks like a parabola. But it is not. It is a more complex curve – the subject of our next activities.

(a) (Refer to Known curve?) If your guess is "the parabola", you are wrong. But do not be disappointed – many great mathematicians including Galileo Galilei made the same wrong guess. In 1669, many years after Galileo, Jungius disproved it. The curve's name is the catenary – derived from the Latin word for "chain." The curve corresponds to a hanging flexible wire or chain supported at its ends and formed by a uniform gravitational force.

The curve can be expressed by the formula

$$y = 0.5(e^{x} + e^{-x})$$



The function is now known as the *hyperbolic cosine*, written as *cosh(x)*,

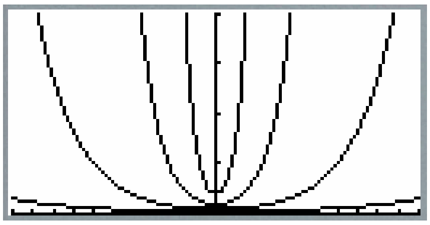
$$\cosh(x) = \frac{(e^{x} + e^{-x})}{2}.$$

Note that the function cosh(x) can be accessed through CATALOG (SHIFT+4).

(b) (Refer to Changing shape) The modification of the curve (similar to that caused by stretching your arms) can be simulated by the change of the parameter a in the formula

$$y = a\cosh(x/a) = 0.5a(e^{x/a} + e^{-x/a})$$

The graph editor contains several modifications of this formula for different values of *a*. Select and unselect them to view the influence of the parameter *a* to it. Because the exponential function e^x grows extremely quickly, we display only a close interval of x = 0 for *a* having the values 0.25, 0.5, 1, and 2.



EXERCISES A

Exercise 2.

What is the relationship between the shape of the hyperbolic cosine curve and the parameter a? For simplicity, use the "rope terminology".

SOLUTIONS to EXERCISES A

Exercise 2.

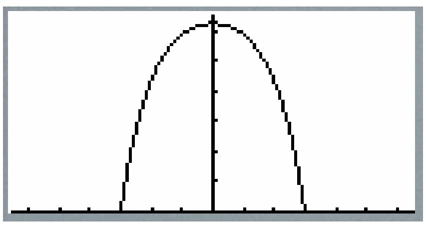
Presuming that the ends of the rope are fixed on a permanent construction the curvature only depends on the length of the rope. The shorter rope and the smaller parameter a: both have the same effect.

(c) (Refer to Internet) To see an unusual application of the catenaries visit the web page http://mathworld.wolfram.com/Roulette.html

The surface made of the catenaries can be used for a smooth motion of "wheels" made of regular polygons. The animation explains and illustrates their unusual movement.

Activity 2 (F10ARCH.g1e):

(a) (Refer to St. Louis Arch) In 1965 in Saint Louis (Missouri, USA), a huge arch was built. It symbolizes an important period of the town history: for almost two centuries, St. Louis used to be the gateway to the West. Its shape has the form of the (inverted) cosine hyperbolic.



The arch curve has been defined using the function

$$y = 693.8597 - 68.7672 \cosh(0.0100333\, x)$$

To view its excellent shape, the parameters of V-WINDOW are set up to the values:

X_{min} = -650

$X_{max} = 650$
$Y_{min} = 0$
$Y_{max} = 650$

These values guarantee the 2:1 proportion between the x and y scale – the proportion that fits best to the screen and displays the curve in the form closest to the reality.

EXERCISES B

***Exercise 1*.**
What is the height of the Arch maximum point from the surface?

***Exercise 2*.**
What is the distance between the "legs" of the Arch?

SOLUTIONS to EXERCISES B

***Exercise 1*.**
The curve is visible above. Evidently, the x-coordinate of the arch maximum is 0 and its y-coordinate is

$$693.8597 - 68.7672 \cosh(0)$$

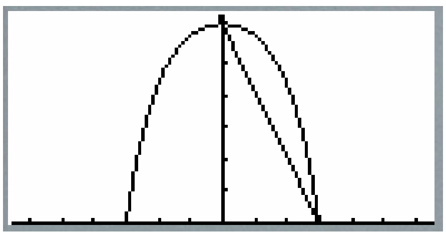
The result is 625.0925 feet i.e. 190.528 meters.

***Exercise 2*.**
The y-coordinate of the ground-touching point is 0. The x-coordinates are the solutions of the equation

$$693.8597 - 68.7672 \cosh(0.0100333\ x) = 0$$

The solutions are -299.226 and 299.226. The distance is about 600 feet i.e. 182 m.

(b) Refer to <u>Straight line</u>) Inside the tube of the Arch, elevators carry the Arch visitors to the top. Let us calculate the length of the trajectory the elevator is making.
To measure the length of the curve is not easy. On the other hand, estimating its length is easier. Let us replace the curve by a straight line between the point where the arch touches the surface and its maximum. Their coordinates are easy to calculate. Look at the above picture:



The line connecting the top and the touching point is defined as y = 625.0925 – (625.0925/299.226)*x

(c) (Refer to Pythagoras) Its length between the two points is the length of the shortest possible lift. Even if the length of the real lift is bigger, we can use it as the first estimation. As it is a hypotenuse of a right-angle triangle, the Pythagoras theorem is appropriate:

$$c = \sqrt{a^2 + b^2} = \sqrt{625.0925^2 + 299.226^2} = 693.020081$$

(d) (Refer to Two lines) The above result is far from the exact length of the curve. On the other hand, it gives us a direction to continue. If we use two straight lines (each covering a half of the distance), the "broken line" formed by them will fit better to the curve. The breaking point will have the coordinates

x = 150

y = 693.8597 - 68.7672 cosh(0.0100333 * 150) = 531.35

Our approximation of the hyperbolic cosine function now consists of two lines:

- From (0, 625.0925) to (150, 531.35) and
- From (150, 531.35) to (299.226, 0)

Again, we can use the Pythagoras theorem to calculate their lengths. They are 176.87 ft and 551.91 ft, respectively. Their total is 728.79.

(e) (Refer to Six lines) We can use a similar principle to achieve even higher precision. Let us split the Arch to 6 section with there horizontal dividing points on x = 0, 50, 100, 150, 200, 250, 299.226 ft.
The result is 738.9. It is rather close to our previously calculated elevator length 728.79 ft. This indicates that our six-section line simulates the curve rather well and we are close to the exact value.

EXERCISES B

***Exercise 3*.**
The Saint Louis Arch function is

y = 693.8597 - 68.7672 cosh(0.0100333 x)

Explain why the coefficient of the cosh is negative.

***Exercise 4*.**
The Saint Louis arch function is

y = 693.8597 - 68.7672 cosh(0.0100333 x)

Explain the role of the constant 693.8597.

Exercise 5.
Use the 12-point division (with 25-feet distances) to calculate the elevator length.

SOLUTIONS to EXERCISES B
Exercise 3.
The values of e^x and e^{-x} are always positive. Their sum is always positive. If the coefficient is also positive, the function has the form of "hanging rope". Its negation inverts the curve "upside down".

Exercise 4.
Without a constant, the functional values would all be negative. The constant "rises the curve above the surface". (The rest of the curve lies under ground.)

Exercise 5.
The result is 739.84 – about one foot more than the 6-point approximation. The further division can hardly lead to much better precision. It means that we can assume the length 740 feet (about 225 meters) as the satisfactory result.

REFERENCES
[T] William V. Thayer: Owner's manual for Saint Louis Arch, !993.
http://www.jug.net/wt/archcgs.htm

[W] http://mathworld.wolfram.com/Catenary.html

