An application to derivatives-A probability problem

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LEVEL
High school or university students who have studied the probability and derivatives.

OBJECTIVES
To use calculator to explore real-life problems in probability and applications to derivatives.

Corresponding eActivity
PROB1.g1e(for Activity1), PROB2.gle(for Activity2)

OVERVIEW
We will see how to explore a probability problem with the use of calculator, which traditionally is difficult. We will solve the problem from graphical, algebraic and analytical point of view.

EXPLORATORY ACTIVITIES
[Note]
We shall use small letter *x* instead of capital X as shown on the calculator throughout the paper.

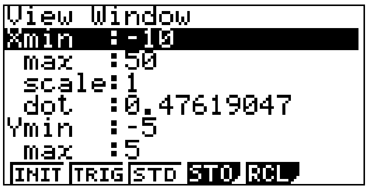
Here we describe activities which represent two special cases from a problem in [YL], page 69.

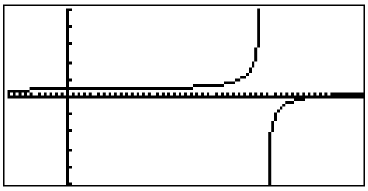
Activity 1: Suppose we have 50 white balls and 50 black balls, which we are going to put some white balls and some black balls into two urns A, and B. <u>The rules are that we have to have at least one ball in each urn and the number of black balls 'doubles' the number of white balls in urn A</u>. A person walks into the room; explain how we can place the white balls and black balls in urn A and urn B respectively so we can
(a) Maximize the probability that this person draws a white ball, and
(b) Minimize the probability that this person draws a white ball.

Solution:
Let *p(x)* be the probability that a single ball drawn at random will be white ball. Thus, we have (when we assume $x \neq 0$)

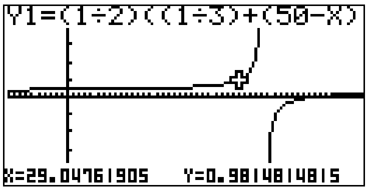
$$p(x)=\frac{1}{2}\left(\frac{x}{x+2x}+\frac{50-x}{100-x-2x}\right)=\frac{1}{6}+\frac{50-x}{200-6x}=\frac{3x-125}{3(3x-100)}.$$

<u>A. Graphical exploration:</u>
(a) We open the <u>*p(x)* graph editor</u> and graph the function *p(x)*. We obtain the following graph by using an appropriate View Window:

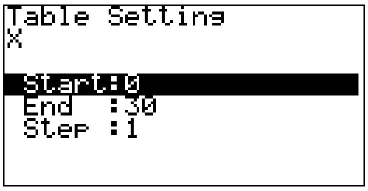




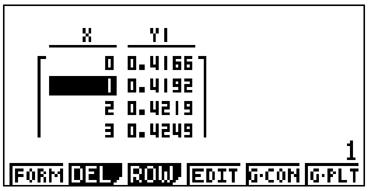
(b) We press [F1][Trace] to explore the probability for the function *p(x)* as follows:

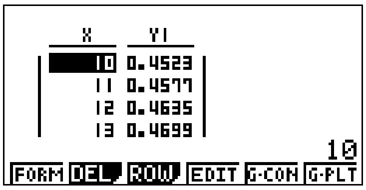


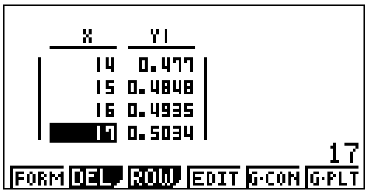
<u>B. Tabular exploration:</u>
(a) Enter the function *p(x)* in the <u>table editor</u> and use the following table setting:

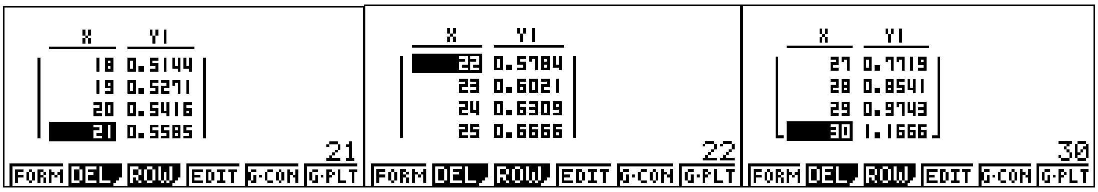


(b) We obtain the following partial list of data.









C. Conjecture:
(a) The minimum happens near x=1 (x=0 is not possible, since the rule says we need to put at least one ball in each urn), or probability of 0.4192. This means that we should put 1 white ball and 2 black balls in urn A and put 49 white balls and 48 black balls in urn B.
(b) The maximum happens near x=29 but this would mean we should put 58 black balls in urn A, which is not suitable. Therefore, the maximum probability happens when we put 24 white balls and 48 black balls in urn A, and 26 white balls and 2 black balls in urn B.

D. Analytical Approach:
(a) If we refer to the graph of *p(x)* above, it seems that *p(x)* has a vertical asymptote. To find the asymptote, we set 3x-100=0 and find x=100/3 to be the vertical asymptote.

(b) It follows from

$$p'(x) = \frac{25}{(3x-100)^2}$$

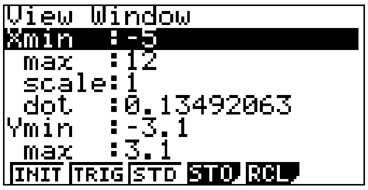
that since $p'(x)>0$ for all x, we conclude that *p(x)* is increasing and the minimum of *p(x)* happens at x=1 and maximum happens at x=24 as we mentioned in section C above.

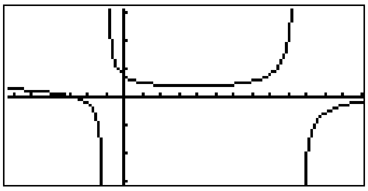
Activity 2: Suppose we have 50 white balls and 50 black balls, which we are going to put some white balls and some black balls into two urns A, and B. The rules are that we have to have at least one ball in each urn and the number of black balls 'squares' the number of white balls in urn A. A person walks into the room; explain how we can place the white balls and black balls in urn A and urn B respectively so we can
(a) Maximize the probability that this person draws a white ball, and
(b) Minimize the probability that this person draws a white ball.

Solution:
Let *p(x)* be the probability that a single ball drawn at random will be white ball. Thus, we have

$$p(x) = \frac{1}{2}\left(\frac{x}{x+x^2} + \frac{50-x}{100-x-x^2}\right).$$

A. Graphical exploration:
(a) We open the graph editor and graph the function *p(x)*. We obtain the following graph by using a proper View Window:

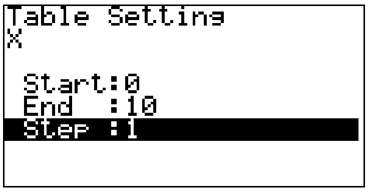




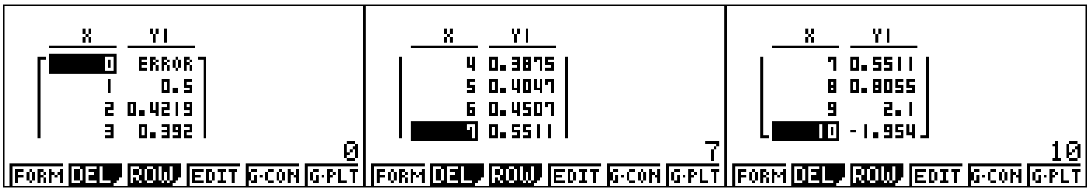
(b) We use F1 [Trace] to explore the probability for the function *p(x)*.

B. Tabular exploration:

(a) Enter the function *p(x)* in the table editor and use the following table setting:



(b) We see the following data.



C. Conjecture:

(a) The minimum happens near x=4, or probability of 0.3875. This means that we should put 4 white balls and 16 black balls in urn A (since the number of black balls is the square of that of white balls in urn A). This implies that there are 46 white balls and 34 black balls in urn B.

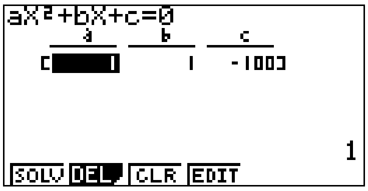
(b) The maximum happens near x=8, but x=8 is not suitable since $8^2 = 64 > 50$. Therefore, the maximum probability happens at x=7, or probability of 0.5511. This means that we should put 7 white balls and 49 black balls in urn A, and 43 white and 1 black balls in urn B.

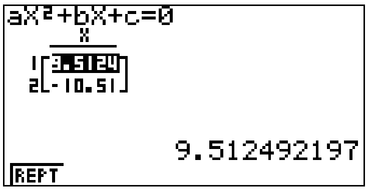
D. Analytical Approach:

(a) If we refer to the graph of *p(x)* above, it seems that p(x) has vertical asymptotes. To find the asymptotes, we observe the following:

$$\frac{x}{x+x^2} + \frac{50-x}{100-x-x^2} = \frac{2x^2-48x-150}{(x+1)(x^2+x-100)}.$$

(b) To find $x^2 + x - 100$ =0, we select the 'Finding roots (Poly Equation strip)' (assuming you are in the eActivity mode) we can use the



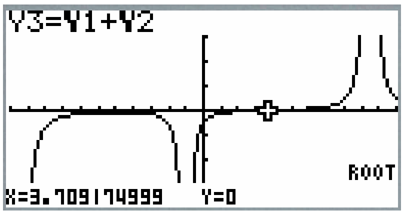


(c) Therefore, the vertical asymptotes are at x=-1,-10.51 and about x=9.512

(d) Since $p(x)=\frac{1}{2}\left(\frac{x}{x+x^2}+\frac{50-x}{100-x-x^2}\right)$, we have

$$p'(x)=\frac{1}{2}\left(\frac{-1}{(x+1)^2}+\frac{-(x^2-100x+50)}{(x^2+x-100)^2}\right)$$

(e) We plot the graph of *y=p'(x)* (open graph of *p'(x)*) and find the root of *y=p'(x)* as follows (press F6 F5 and ROOT):



- Notice that since p'(x) changes signs from negative to positive at x=3.709174999, *p(x)* has a minimum at x=4 (since p(3)= 0.3920454545 and p(4)= 0.3875), which is consistent with our conjecture.
- Notice that *p'(x)*>0 in (3.709174999, 9.512), this implies that p(x) is increasing in such interval and we conclude that the maximum of *p(x)* occurs at x=7 (since x=8 and 9 are not suitable, please refer this to section C Conjecture (b) above).

EXERCISES

Exercise 1. Redo the activity 1 above by using n=100.
Exercise 2. Redo the activity 2 above by using n=100.

SOLUTIONS

Solution to Exercise 1.
The minimum probability happens at x=1, which means you place 1 white ball and 2 black balls in urn A and 99 white balls and 98 black balls in urn B. The maximum probability happens at x=49, which means you place 49 white balls and 98 black balls in urn A and 51 white and 2 black balls in urn B.

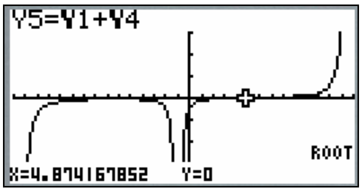
Solution to Exercise 2.
(a) We define $q(x)=\frac{1}{2}(\frac{x}{x+x^2}+\frac{100-x}{200-x-x^2})$, which can be simplified as

$$q(x) = \frac{x^2 - 49x - 150}{(x+1)(x^2 + x - 200)}.$$

(b) There are two vertical asymptotes for *q(x)*: x=-14.6509717 and x=13.6509717.

(c) We find $q'(x) = \frac{-1}{2}\left(\frac{1}{(x+1)^2} + \frac{x^2 - 200x + 100}{\left(x^2 + x - 200\right)^2}\right).$

(d) We plot *y=q′(x)* and find its root as follows:



(e) Since *q(4)*=0.36667 and *q(5)*=0.3627451, the minimum probability for *q* is at x=5. This means the probability of drawing a white ball is by placing 5 white balls and 25 black balls in urn A; and 95 white ball and 75 black balls in urn B.
(f) Since *q* is increasing in (4.874167852, 13.6509717), and x=10 or 11 is not suitable, we conclude that the maximum probability of drawing a white ball is when we place 9 white balls and 81 black balls in urn A, 91 white balls and 19 black balls in urn B.

REFERENCE

[YL] W.-C. Yang and J. Lewin, *Exploring Mathematics with Scientific* Notebook, Springer Verlag, 1998. ISBN 981-3083-88-3.

