Mathematics with a Scientific Calculator

Casio *fx*-991ES PLUS

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Table of Contents

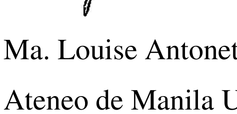
Preface

The advent of different types of technologies inevitably influences the teaching and learning of mathematics. A challenge to mathematics educators is how to use these technologies to improve their competencies and increase their efficiencies as teachers to make the mathematical lives of their students better.

The emergence of modern scientific calculators such as the Casio 991 ES Plus Scientific Calculator with extensive numeric capabilities is paving way for educators to rethink the curriculum and to develop instructional and learning materials requiring more in-depth analysis and with more emphasis on real world problems. There is a wide range of mathematical opportunities that are made accessible with this calculator given its tabular and list capabilities, functionalities related to solutions of equations, matrix and statistical calculations. The use of the numerical and tabular features can be maximized to facilitate graphical interpretation of functions and data analysis; and the numeric solving capabilities of the calculator can be used to find solutions to real world problems and perform mathematical exploratory tasks.

This book is a collection of lessons and activities covering a wide range of topics designed to address the needs of teachers and students in studying mathematics equipped with a modern scientific calculator, such as the Casio 991 ES Plus. The lessons given here suggest the kinds of mathematical problems that may be addressed using the calculator, given the understanding of the mathematical ideas involved. The examples and situations presented here provide a sample of the many ways we can maximize the benefits of a modern scientific calculator to learn mathematics efficiently.

The material included in this book is a result of my involvement in the various ES Scientific Calculator projects carried out with Casio Computer Co., Inc. through the years, in collaboration with several mathematics educators. I wish to thank Casio Computer Co., Inc. for their commitment and support and for sharing in our vision to improve the state of mathematics throughout the world. My thanks also go to Casio Philippines under the leadership of Mary C. Chan, for making this material available to the teachers and students in the Philippines.



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Ateneo de Manila University, March 2013

1. Algebra

Proportion

An equation that equates two ratios is called a proportion. Thus, if $b \neq 0$ and $d \neq 0$ then $\frac{a}{b} = \frac{c}{d}$ is a proportion.

Students can understand and appreciate more the concept of proportion through applications carried out with the aid of a scientific calculator.

Activity 1. Solve the proportion $\frac{x+2}{8} = \frac{2x-3}{7}$.

Solution:

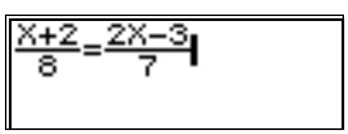
We enter the expression $\frac{x+2}{8} = \frac{2x-3}{7}$ in the computational mode of the calculator and solve the equation.

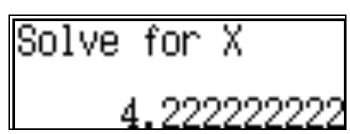
[Operations]

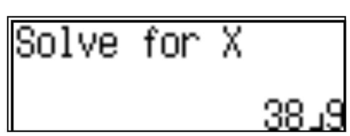
- Press Mode. Select 1: COMP — [MODE] [1]
- Enter Quotient — [▭/▭]
- Enter $\frac{x+2}{8}$ — [ALPHA] [)] (X) [+] [2] [▼] [8] [▶]
- Enter equal "=" — [ALPHA] [CALC] (=)
- Enter Quotient — [▭/▭]
- Enter $\frac{2x-3}{7}$ — [2] [ALPHA] [)] (X) [−] [3] [▼] [7] [▶]
- Solve the equation — [SHIFT] [CALC] (SOLVE)

Press the equal sign to get the approximate answer [=]

The exact answer is obtained by pressing [S⇔D]







Activity 2. Equate the cross products of the given proportion in activity 1. Then solve for *x*. Do you

obtain the same result when solving the proportion $\frac{x+2}{8} = \frac{2x-3}{7}$?

Solution:

If we equate the cross products of the proportion $\frac{x+2}{8} = \frac{2x-3}{7}$, we obtain the equation 7(x + 2) = 8(2x -3).

[Operations]

- Press Mode. Select 1: COMP — [MODE] [1]
- Enter 7(x+2) — [7] [(] [ALPHA] [)] (X) [+] [2] [)]
- Enter equal "=" — [ALPHA] [CALC] (=)
- Enter 8(2x-3) — [8] [(] [ALPHA] [)] (X) [−] [3] [)]
- Solve the equation — [SHIFT] [CALC] (SOLVE)

The exact answer is obtained by pressing [S⇔D]

7(X+2)=8(2X-3)	Solve for X 4.222222222

Observe that the cross products of any proportion are equal. That is for $b \neq 0$ and $d \neq 0$, the proportion $\frac{a}{b} = \frac{c}{d}$ is equivalent to $ad = bc$. Note that if we multiply both sides of the equation by bd, we obtain

$$(bd)\frac{a}{b} = \frac{c}{d}(bd) \text{ which implies } ad = bc.$$

Consider the following application of proportion to a real world problem.

Example. In a sample of ballots, it is found that 9 ballots out of 5,100 are incorrectly marked and must be thrown out. How many incorrectly marked ballots would be expected in a state where 2,900,000 ballots are cast?

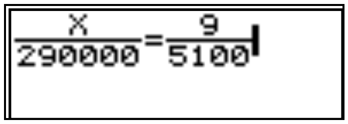
Solution:

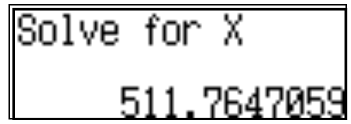
Let x represent the number of incorrectly marked ballots corresponding to the 2,900,000 ballots that were cast.

Note that 9 ballots out of the 5,100 were incorrectly marked. Hence we have the proportion

$$\frac{x}{290000} = \frac{9}{5100}$$

We use the same calculator commands given in the previous activity. The screen dumps are as follows:





The answer would be approximately 512 ballots.

EXERCISES

1.Solve the proportion $\frac{x-3}{4x}=\frac{2}{x+7}$.

Solution:

For a quadratic proportion, we first determine the cross products and solve for the resulting quadratic equation.

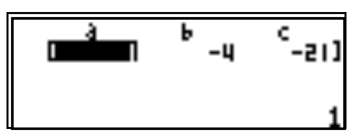
We obtain (x-3)(x+7) = 2(4x) or equivalently, x^2-4x-21=0.

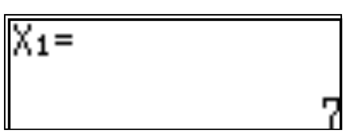
The following calculator commands will be used:

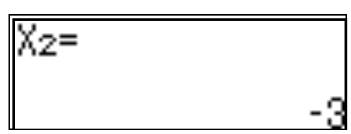
[Operations]

- Press Mode. Select 5: EQN [MODE] [5]
- Select quadratic equation [3]
- Enter coefficients of the equation in the table then press [=].

Example: Enter 1,-4,-21 [1] [=] [(–)] [4] [=] [(–)] [2] [1] [=]







We get two answers 7 and -3.

If we solve the equation while in the computational mode of the calculator, we only get one answer. Thus for quadratic and cubic proportions, we use the equation mode of the calculator.

2. A small plane flies 450 miles with the wind from Memphis, Tennessee to Oklahoma City, Oklahoma. In the same amount of time, the plane flies 345 miles against the wind from Oklahoma City to Little

Rock, Arkansas. If the wind speed is 32 miles per hour, find the speed of the plane in still air.

Solution:

Let x represent the speed of the plane in still air.

We have the following chart that describes and relates the distance, rate and time corresponding to the plane traveling with and against the wind. Recall that distance equals rate and time, that is, $D = RT$.

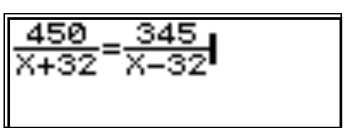
	DISTANCE *D*	**RATE *R***	**TIME *T*= *D/R***
Against the wind	345 miles	x -32 miles/hr	$\frac{345}{x-32}$
With the wind	450 miles	x+32 miles/hr	$\frac{450}{x+32}$

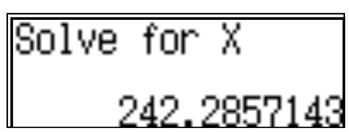
Now, the plane travels with the wind for the same amount of time as it travels against the wind, so the proportion is

$$\frac{450}{x+32} = \frac{345}{x-32}$$

We obtain the answer $x \approx 242.29$ miles per hour. The plane's speed in still air is about 242 miles per hour. Given that the plane's speed in still air is this rate, then the corresponding time is approximately 1.64 hours.

Using the computational mode of the calculator we get the following screen dumps:





Exploring Higher Order Polynomial Functions

In this section we use the tabular function of the calculator to study properties pertaining to local maxima and minima of polynomial functions.

If the point ($a,f(a)$) is the highest point on the graph of a function f within an open interval I containing a then $f(a)$ is a local maximum value of f. On the other hand if the point ($b, f(b)$) is the lowest point on the graph of a function f within an open interval I containing b, then $f(b)$ is a local minimum value of f.

ACTIVITIES

Note: Unless otherwise specified, we choose MATH mode in the SET UP menu, using
[SHIFT] [MODE] [1] (MthIO)

Activity 1. Consider the following polynomial function of degree three:

$$f(x) = (x-2)(x-4)(x-5)$$

Determine how many local extrema the given function has.

Solution:

From the factored form of $f(x)$, we can see that 2,4,6 are zeroes or roots of the given function. This means the graph of $f(x)$ crosses the $x-$ axis at the x values 2,4,5.

We use the table function of the scientific calculator to further analyze the graph of $f(x)$.
The operations below show how to enter the function for tabulation.

[Operations]

- Enter Table mode — [MODE] [7]
- Enter $(x-2)$ — [(] [ALPHA] [)] (X) [−] [2] [)]
- Enter $(x-4)$ — [(] [ALPHA] [)] (X) [−] [4] [)]
- Enter $(x-5)$ — [(] [ALPHA] [)] (X) [−] [5] [)]
- Press = to finish — [=]

The next step is to decide which values to tabulate, as the calculator screen shows “Start?”. Choose a start value for x, an end value for x and the increment (step) between values. We choose some x values on the intervals (2,4), (4,5).

First, we look at x values on the interval (2,4). We will use the interval from $x = 2$ to $x = 4$ with step 0.1.Continue the operations below.

[Operations]

- Enter Start value [2] [=]
- Enter End value [4] [=]
- Enter Step value [•] [1] [=]
- Explore table of values [▶] [▲] [▼] [▼] [▼] [▼] etc

While exploring values in the table, observe that when you navigate in the right column; along the $f(x)$ values, exact functional values will appear in the bottom right of the calculator screen.

By generating the values of the function from the table, we have the following observations:

- $f(2)=0$, $f(2.1)=0.551$, $f(2.2)=1.008$, $f(2.3)=1.377$, $f(2.4)=1.664$, $f(2.5)=1.875$, $f(2.6)=2.016$, $f(2.7)=2.093$, $f(2.8)=2.112$, $f(2.9)=2.079$, $f(3)=2$, $f(3.1)=1.881$, $f(3.2)=1.728$, $f(3.3)=1.547$, $f(3.4)=1.344$, $f(3.5)=1.125,\ldots$, $f(4)=0$
- The values of the function are increasing from $x = 2$ to $x = 2.8$
- The values of the function are decreasing immediately after $x = 2.8$ to $x = 4$

This implies there is a local maximum value somewhere between $x = 2.8$ to $x = 2.9$

Next, we look at x values in the interval $(4,5)$. This time, we will use the interval from $x = 4$ to with $x = 5$ step 0.1. We use the following operations:

[Operations]

- Return to the function [AC] [=]
- Enter Start value [4] [=]
- Enter End value [5] [=]
- Enter Step value [•] [1] [=]
- Explore table of values [▶] [▼] [▼] [▼] [▼] etc

From the table of values generated, we record the following observations:

- $f(4)=0$, $f(4.1) = -0.189$, $f(4.2) = -0.352$, $f(4.3)= -0.483$, $f(4.4)=-0.576$, $f(4.5)=-0.625$, $f(4.6)=-0.624$, $f(4.7)=-0.567$, $f(4.8) = -0.448$, $f(4.9) = -0.261$, $f(5)=0$
- The values of the function are decreasing from $x = 4$ to $x = 4.5$
- The values of the function are increasing immediately after $x = 4.5$ to $x = 5$

This implies there is a local minimum value somewhere between $x = 4.5$ to $x = 4.6$.

By choosing smaller intervals(smaller steps), further investigation can be done with regards to finding the approximate local maximum or minimum value of the function.

Consider for instance, for this example, the local minimum value. We can consider exploring further the values between $x = 4.5$ to $x = 4.6$ with the increment of 0.01.

[Operations]

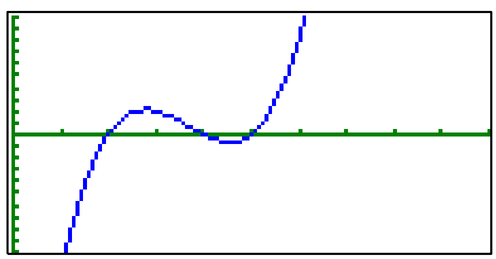
- Return to the function — [AC] [=]
- Enter Start value — [4] [•] [5] [=]
- Enter End value — [4] [•] [6] [=]
- Enter Step value — [•] [0] [1] [=]
- Explore table of values — [▶] [▼] [▼] [▼] [▼] etc

We have the following results:

f(4.5) = -0.625, *f*(4.51) = -0.627, *f*(4.52) = -0.628, *f*(4.53) = -0.63, *f*(4.54) = -0.63, *f*(4.55) = -0.631, *f*(4.56) = -0.63, *f*(4.57) = -0.629, *f*(4.58) = -0.628,etc.

We approximate the local minimum value to be -0.631 at x=4.55.
Using a similar process, we can approximate the local maximum value to be 2.11 at x=2.8
Thus, there are two local extrema for this function.

Drawing the graph by hand, the graph should look like the following:



Since the polynomial is of degree odd and leading coefficient is positive, the graph takes the above shape. Indeed there are two local extrema values.

Activity 2. Determine the number of local extrema for the following functions:

a. $f(x) = (x+3)x(x-1)(x-3)$
b. $g(x) = (x+4)(x+1)(x-1)(x-3)(x-5)$

What generalization can you give regarding the number of local extrema of the following function:
$f(x) = (x-x_1)(x-x_2)\ldots(x-x_n)$?

Solution:

We follow the approach given in the previous activity.

a. We check in the intervals (-3,0),(0,1) and (1,3).The local minimum value is approximately -30.01 at x = -2.02.There is another local minimum value, approximately -11.1 at x = 2.29. The local maximum

value is approximately 2.19 at x =0.49.
There are three local extrema.

b. Checking at the intervals (-4,-1),(-1,1),(1,3) and (3,5), we have the following results:
local minimum value of -61.1 at x = -0.13; -132.15 at x = 4.3
local maximum value of 384.24 at x = -3.03;54.3 at x =2.1
There are four local extrema.

Our generalization is that the function $f(x) = (x - x_1)(x - x_2)\ldots(x - x_n)$ has n-1 local extrema.
If f has a local extrema at x = c then c is found in either of the intervals $(x_1, x_2), (x_2, x_3), \ldots (x_{n-1}, x_n)$

EXERCISES

1. Find the local extrema of $f(x) = (x+4))(x+1)(x-1)(x-3)(x-5)+7$.

Solution:

The results we have of activity 2, indicate we have the following calculations referring to $g(x) = (x+4))(x+1)(x-1)(x-3)(x-5)$

local minimum value of -61.1 at x = -0.13; -132.15 at x =4.3
local maximum value of 384.24 at x = -3.03; 54.3 at x =2.1

Now, to determine the local extrema of $f(x) = (x+4))(x+1)(x-1)(x-3)(x-5)+7$
we add 7 to the above calculations.

Thus we obtain

local minimum value of -61.1+7 = -54.1 at x = -0.13; -132.15+7= -125.15 at x =4.3
local maximum value of 384.24 +7=391.24 at x = -3.03;54.3+7=61.3.3 at x =2.1

2. Is it possible for a third- degree polynomial to have exactly one local extremum?

Solution:

No, it is not possible. If the 3rd degree polynomial had one local extremum, then from our previous activity, this must lie in (a,b) where a,b are zeroes or roots of the third degree polynomial. However, a third degree polynomial can only have 1 or 3 roots, since complex zeroes come in pairs.

3. How many local extrema does the polynomial $f(x) = x^3 - 4x$ have? How about $f(x) = x^3 + 4x$?
If $a > 0$, how many local extrema does each of the polynomials $f(x) = x^3 - ax$ and $f(x) = x^3 + ax$ have?

Solution:

For $f(x) = x^3 - ax = x(x^2 - a) = x(x - \sqrt{a})(x + \sqrt{a})$ there are two local extrema in the intervals

$(-\sqrt{a},0),(0,\sqrt{a})$.

There are none for $f(x)=x^3+ax=x(x^2+a)$. There is only one real zero of the function.

Data Analysis and Modeling

The purpose of the activities given in this section is to use functions to mathematically model relationships observed in the real world with the use of the scientific calculator.

Data analysis, usually carried out by fitting curves to data and discussing mathematical models, allows students to understand mathematical patterns occurring in natural phenomena and makes them gain insight into using these models to make predictions. There are many types of functions that can be used to model relationships, for instance, linear, quadratic, exponential and logarithmic functions. Technology is a very helpful aid to students in helping them select the appropriate function to model a given set of data.

Linear Regression

Linear regression is a procedure that can be used to model a set of data using a linear function. The correlation coefficient *r* is a statistical measure of how well the data lies along the regression line or how well the two variables are correlated. The coefficient of correlation serves as a quantitative basis of judging how good a model the linear function is for predictive purposes.

ACTIVITY

Determine the linear function that best models the given data. Compute and interpret the coefficient of correlation *r*.

a. Data on the number of vehicles produced in the world (in millions) since 1950:

Year, x	**Number of Vehicles produced in the world, y (in millions)**
1950,0	10.6
1960,10	16.5
1970,20	29.4
1980,30	38.6
1990,40	48.6
1995,45	50.0

Source: *Wall Street Journal Almanac*, 1999

b. Data on the relative abundance of mosquitoes x, measured by mosquito positive rate and flow rate y, measured in terms of percentage of maximum flow, of canal networks in Saga City, Japan:

flow rate %, x	0	10	40	60	90	100
mosquito positive rate %, y	22	16	12	11	6	2

c. University students' data, consisting of math SAT scores x and math placement test scores y which the students have taken before entrance to the university and registration, respectively.

Math SAT scores x	**Placement TEST scores** y	**Math SAT scores** x	**Placement Test scores** y
540	20	580	7
510	16	680	15
490	10	560	8
560	8	560	13
470	12	500	14
600	11	470	10
540	11	550	9
610	9	620	8

Solution:

We calculate the linear function (regression line) and the correlation of coefficient for each of the given data using the following calculator keystrokes:

[Operations]

- Press Mode. Select 3: STAT [MODE] [3]
- Select 2: A+ BX Linear Regression [2]
- Enter each of the data in the table then press [AC].
- Calculate the regression line A+BX

  Calculate the Slope B [SHIFT] [1] [5] [2] [=]

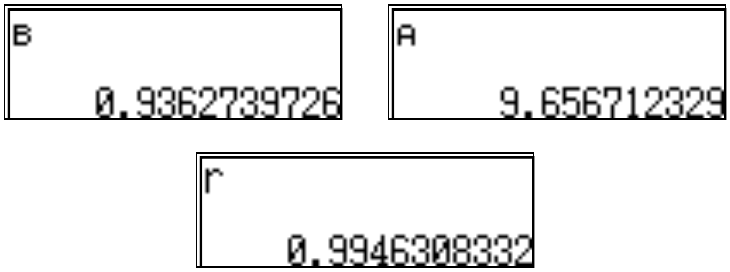
  Calculate the Y-intercept A [SHIFT] [1] [5] [1] [=]
- Calculate the correlation coefficient, *r* [SHIFT] [1] [5] [3] [=]

Note: The correlation coefficient, the slope and the y-intercept is to be calculated one at a time. After finding the correlation coefficient, press AC, calculate slope, then press AC again to calculate *y*-intercept.

- Whenever we want to refer to the same data again, [SHIFT] [1] [2]
  The data will be kept in the calculator's memory.

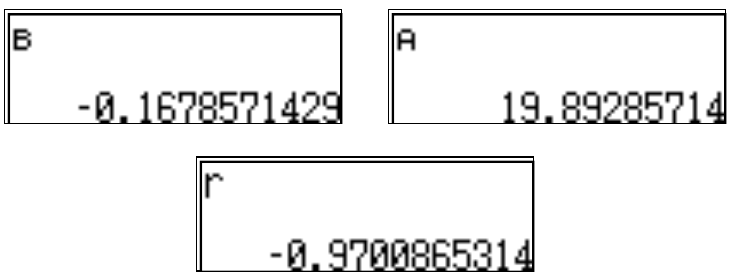
a. The ascending straight line, the regression line $A + Bx$ for the given data is 9.6567 + 0.9363*x*.

We obtain a coefficient of correlation r = 0.9946 which indicates a very good linear correlation. This means that the points on the regression line give a good approximation of the data points. The line can be used as a good predictor. A positive value of r indicates that the regression line has a positive slope. A positive linear relationship exists, the points fall approximately in an ascending straight line from left to right, and both x and y values increase at the same time. As the years pass by, from the year 1950, the number of vehicles produced increases.

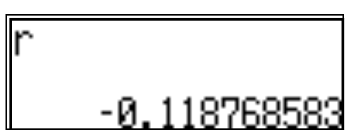


b. For the given data, the coefficient of correlation is $r = -0.97$ which also indicates a very high linear correlation; the regression line should be a good predictor. However, a negative value of r indicates that the regression line has a negative slope. A negative linear relationship exists, the points fall approximately in a descending straight line from left to right. As the x values are increasing then the y values are decreasing. As % of flow rate of water in canals is increasing, the % of mosquitoes present is decreasing.

The descending straight line, the regression line $A + Bx$ for the given data is $19.8929 - .1679x$.



c. The last set of data do not fit a line as seen from the correlation coefficient we obtained, $r = -0.1188$. There is almost no linear correlation between the data points, that is, there is no linear relationship between the Math SAT scores and Math Placement test scores. A linear function cannot be used to model the given data.



EXERCISES

1. Use the regression line given in item a to estimate the year when the number of vehicles produced is 70 million.

2. Use the linear model in question b to estimate the mosquito positive rate if the canal flow is 80% maximum.

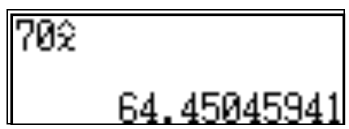
Solution:

1.We let $y = 70$ and we calculate for the value of x using the linear model as the predictor using the calculator.

[Operations]

- Press Mode. Select 3: STAT — MODE 3
- Select 2: A+ BX Linear Regression — 2
- Enter each of the data in the table then press AC.
- Enter 70 — 7 0
- Calculate the x value given the y value — SHIFT 1 5 4 =

We obtain $x = 64.45$, which is approximately 65. Thus, 70 million vehicles were produced at about the year 2015.

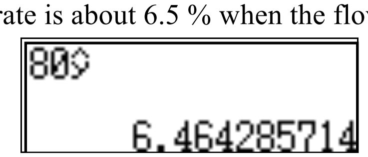


2. This time, we let $x = 80$ and we calculate the value of y using the linear model as the predictor.

[Operations]

- Press Mode. Select 3: STAT — MODE 3
- Select 2: A+ BX Linear Regression — 2
- Enter each of the data in the table then press AC.
- Enter 80 — 8 0
- Calculate the y value given the x value — SHIFT 1 7 5 =

We obtain $y = 6.464$. Thus, mosquito rate is about 6.5 % when the flow rate of canal network is 80%.



Quadratic Regression

In quadratic regression, a quadratic function is used to model the given data. Let us consider the following example:

ACTIVITY

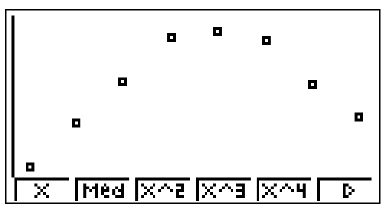
A physicist throws a ball at an inclination of 45° to the horizontal. The following data represent the height of the ball, *y* feet, at the instant it has traveled a distance of *x* feet horizontally.

Distance	Height
10	14
20	30
30	45
40	61
50	64
60	60
70	44
80	32

a. Plot the data on the *x-y* axis. What relationship exists between the variables?
b.What is the function of best fit?
c.Approximate the height of the ball at the instant it has traveled 55 feet horizontally.

Solution:

a. If we plot the data we get a curve as follows:



We see that a quadratic relationship exists between the *x* and the *y* values since the scatter plot suggests a parabolic motion.

b. Since a quadratic relationship exists between the variables, let us perform quadratic regression to the data points.

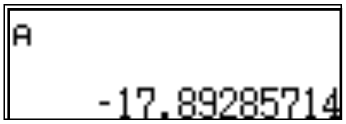
[Operations]

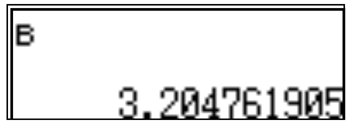
- Press Mode. Select 3: STAT [MODE] [3]
- Select 3: Quadratic Regression _ + CX^2 [3]
- Enter each of the data in the table- *x* values in first column, *y* values in the second column. After all of the data is entered, press [AC].
- Calculate the regression curve A +BX + CX^2

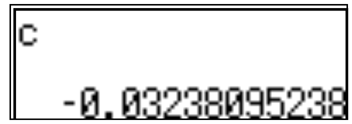
  Calculate the constant term A [SHIFT] [1] [5] [1] [=]

  Calculate the coefficient of X, B [SHIFT] [1] [5] [2] [=]

  Calculate the coefficient of X^2, C [SHIFT] [1] [5] [3] [=]





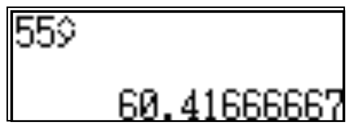


The quadratic function that best fits the data is $y = -0.03x^2 + 3.20x - 17.89$.

c. Let us now approximate the height of the ball at the instant it has traveled 55 feet horizontally.

[Operations]

- Enter 55 [5][5]
- Calculate the y value given the x value [SHIFT][1][5][6][=]



The height of the ball is approximately 60 feet.

EXERCISE

We suggest that students investigate more examples using a quadratic function as a mathematical model. A helpful guide question would be: How can one determine if a given data can be modeled by a quadratic function? Usually, when data points fall at a steady rate, a linear function $y = mx + b$ is used, $m < 0$. When data points rise at a steady rate, the linear function $y = mx + b$ is used, $m > 0$. However, data points that show indication of a rise and a then fall, usually with a maximum point, suggest that a quadratic function $f(x) = ax^2+bx+c$, $a < 0$ might fit the data. On the other hand, when the data points fall and then rise, the quadratic function, $f(x) = ax^2+bx+c$, $a > 0$ could be used to model the data.

Explorations on Fibonacci Numbers

One of the most famous problems in mathematics is the following:

A man put a pair of rabbits in a cage. During the first month the rabbits produced no offspring, but each month thereafter produced one new pair of rabbits. If each new pair thus produced reproduces in the same manner, how many pairs of rabbits will be there at the end of one year?

The solution of this problem leads to a sequence of numbers known as the Fibonacci sequence.

The following are the first 13 terms of the Fibonacci Sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233.$$

If F_n represents the Fibonacci number in the n th position in the sequence, then

$$F_1 = 1,\ F_2 = 1 \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 3$$

In this section we give some exploratory activities on Fibonacci numbers.

ACTIVITIES

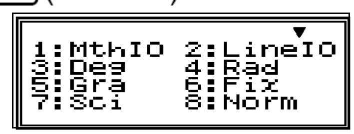
Activity 1. Consider the quotients of successive Fibonacci numbers. What pattern emerges?

Solution:

We use the calculator to solve the quotients.

[Operations]

- Select SHIFT SETUP — SHIFT MODE (SETUP)



- Select LineIO — 2
- Press Mode. Select 1: COMP — MODE 1

To solve a quotient, example, 5÷3

- 5÷3 — 5 ÷ 3 =

We obtain $5 \div 3 = 1.6\bar{6}$.

The quotients of successive Fibonacci numbers(13 terms) are as follows:

$$\frac{1}{1} = 1,\ \frac{2}{1} = 2,\ \frac{3}{2} = 1.5,\ \frac{5}{3} = 1.6\bar{6}, \frac{8}{5} = 1.6,\ \frac{13}{8} = 1.625,$$
$$\frac{21}{13} \approx 1.615384615, \frac{34}{21} \approx 1.619047619, \frac{55}{34} \approx 1.617647059$$
$$\frac{89}{55} \approx 1.618181818,\ \frac{144}{89} \approx 1.617977528, \frac{233}{144} \approx 1.618055556$$

The quotients are approaching some "limiting value" close to 1.618.

Activity 2. a. Let x_1, x_2 be the positive and negative solutions respectively of the equation $x^2 - x - 1 = 0$. Calculate $x_1,\ x_2$.
b. What similarity do you notice between $x_1,\ x_2$?
c. Compare x_1 with the limiting value in Activity 1. What do you observe?

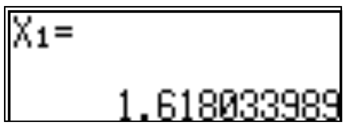
d. Calculate $\frac{1}{x_1}$. What similarity do you notice about x_1 and $\frac{1}{x_1}$?

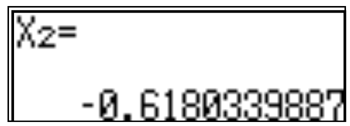
Solution:

a. To calculate x_1, x_2, we use the following calculator commands:

[Operations]

- Press Mode. Select 5 EQN [MODE] [5]
- Select $ax^2 + bx + c = 0$ [3]
- Enter coefficients a=1,b =c = -1 [1] [=] [(–)] [1] [=] [(–)] [1] [=] [=]





We obtain $x_1 = 1.618033989$ and $x_2 = -0.6180339887$

b. After the decimal points the digits of x_1 and x_2 are almost the same.

c. x_1 is the limiting value of the quotients in Activity 1.

d. The following calculator commands are suggested to obtain $\frac{1}{x_1}$ from item 2a:

[Operations]

- After obtaining $x_1 = 1.618033989$, store this in memory using Ans [Ans]
- Press Mode. Select 1: COMP [MODE] [6]
- Access $x_1 = 1.618033989$ [Ans]
- Press x^{-1} [x^{-1}] [=]

The answer we obtain is $x_1^{-1} = .6180339887$. After the decimal points the digits of x_1 and $\frac{1}{x_1}$ are the same.

The exact solutions of the equation $x^2 - x - 1 = 0$ are $x_1 = \frac{1+\sqrt{5}}{2}$ and $x_2 = \frac{1-\sqrt{5}}{2}$. $x_1 = \frac{1+\sqrt{5}}{2}$ is called the **golden ratio**.

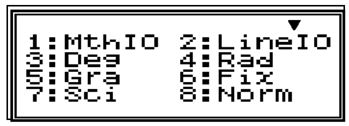
Activity 3. Generate the sequence $\frac{1}{\sqrt{5}}\left\{\left[\frac{1+\sqrt{5}}{2}\right]^n - \left[\frac{1-\sqrt{5}}{2}\right]^n\right\}$ for n=1 to 13. What do you observe regarding the sequence generated?

Solution:

We generate the sequence using the table feature of the calculator as follows:

[Operations]

- Select SHIFT SETUP — [SHIFT] [MODE] (SETUP)

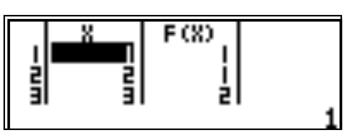


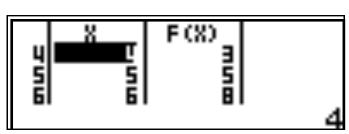
- Select MthIO — [1]
- Press Mode. Select 7: TABLE — [MODE] [7]

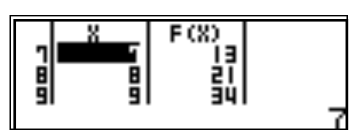
Enter $\frac{1}{\sqrt{5}}\left\{\left[\frac{1+\sqrt{5}}{2}\right]^n - \left[\frac{1-\sqrt{5}}{2}\right]^n\right\}$ in f(x) using n as x as follows:

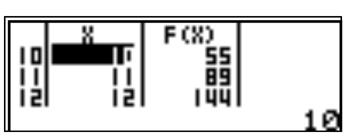
- Access quotient sign — [▭/▭]
- Enter $1, \sqrt{5}$ — [1] [▼] [√▪] [5] [▶] [▶]
- Enter (— [(]
- Enter $\left(\frac{1+\sqrt{5}}{2}\right)$ — [(] [▭/▭] [1] [+] [√▪] [5] [▶] [▼] [2] [▶] [)]
- Enter exponent x — [$x^▪$] [ALPHA] [)] (X) [▶]
- Enter - $\left(\frac{1+\sqrt{5}}{2}\right)$ — [−] [(] [▭/▭] [1] [−] [√▪] [5] [▶] [▼] [2] [▶] [)]
- Enter exponent x — [$x^▪$] [ALPHA] [)] (X) [▶]
- Enter) — [)] [=]
- Enter start value 1 — [1] [=]
- Enter end value 13 — [1] [3] [=]
- Enter step value 1 — [1] [=]

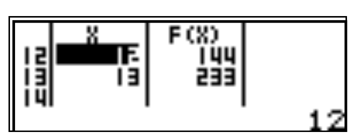
We obtain the following table:











The sequence generated is the Fibonacci sequence.

The form $\frac{1}{\sqrt{5}}\left\{\left[\frac{1+\sqrt{5}}{2}\right]^n - \left[\frac{1-\sqrt{5}}{2}\right]^n\right\}$ is called the Binet form of the n th Fibonacci number.

EXERCISES

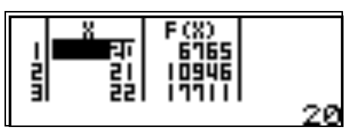
1. Use the Binet form to calculate the n th Fibonacci number for. $n = 20, \ldots, 25$

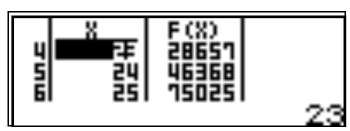
Solution:

[Operations]

- From the table generated in previous exercise, enter AC then equals [=]
- Enter start value 20 [2] [0] [=]
- Enter end value 25 [2] [5] [=]
- Enter step value 1 [1] [=]

We obtain the following table:





We obtain 6765, 10946, 17711, 28657, 46368 and 75025.

2. Another Fibonacci-type sequence that has been studied by mathematicians is the **Lucas sequence**.

The first ten terms of the Lucas sequence are

1,3,4,7,11,18,29,47,76,123

a. What is the eleventh term of the Lucas Sequence?
b. The first term of the Lucas sequence is 1. Add the first and third terms. Record your answer. Now add the first, third and fifth terms and record your answer. Continue this pattern each time adding another term that is in an odd position in the sequence. What do you notice about all of your results?
c.The second term of the Lucas sequence is 3. Add the second and fourth terms. Record your answer. Now add the second, fourth and sixth terms and record your answer. Continue this pattern each time adding another term that is in an even position in the sequence. What do you notice about all of your sums?
d.Make an observation regarding the quotients of successive terms of the Lucas sequence, similar to exploratory task 1. Make a conjecture about what is happening.

Solution:

a. The eleventh term of the Lucas sequence is 76+123=199.

b. We have the following sums:

1+4 = 5; 1+4+11=16; 1+4+11+29=45; 1+4+11+29+76=121
Each sum is 2 less than a Lucas number.

c. We obtain the following:

3+7=10; 3+7+18=28; 3+7+18+47=75; 3+7+18+47+123= 198
Each sum is 1 less than a Lucas number.

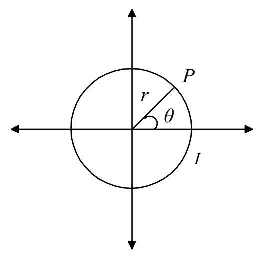
d.The quotients are approaching some "limiting value" close to 1.6 as seen below:

$$\frac{3}{1} = 3,\ \frac{4}{3} = 1.3\bar{3},\ \frac{7}{4} = 1.75,\ \frac{5}{3} = 1.6\bar{6},\ \frac{11}{7} = 1.571428571,$$
$$\frac{18}{11} = 1.636363636,\ \frac{29}{18} \approx 1.61\bar{1},\ \frac{47}{29} \approx 1.620689655$$

2. Trigonometry

Radian Measure

Given an angle whose vertex is at the center of the circle, a **radian** is defined to be the angle that intercepts an arc equal to the radius of the circle. In the figure below, the angle θ is subtended by the arc IP of the circle. If the length of IP is equal to the radius r of the circle, then θ has a measure of 1 radian.

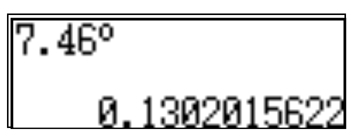


Up to a certain extent, discussion of angles is focused on the degree as the unit of angular measure. However, some technical problems require angles to be measured in radians. Some real world applications of radian measure include the length of a circular arc, circular sector, angular velocity and latitude problems.

A calculator can be a very helpful tool in carrying out calculations involving radian, degree measure and handling angle conversions. Some real world applications of radian measure such as the length of a circular arc, the area of a circular sector and latitude problems will be solved in this section using the calculator.

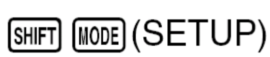
To obtain angle calculations in radians(respectively, degrees), we first specify using [SHIFT SET UP] that the angle will be in radians(respectively, degrees).

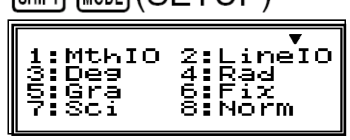
When in radian mode, angles in degrees are converted to radians by entering the degree symbol “o” after the angle measurement. For instance, 7.46 degrees in radians, is approximately 0.1302015622.



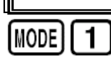
[Operations]

- Press SHIFT SET UP

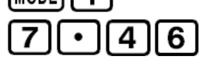




- Select 4: Rad
- Press Mode. Select 1: COMP



- Enter 7.46

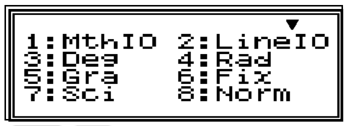


- Enter degrees “o”

On the other hand, when in degree mode, angles in radians are converted to degrees by entering the radian symbol "r". For example $\pi/4$ in degrees is 45.

[Operations]

- Press SHIFT SET UP — SHIFT MODE (SETUP)



- Select 3: Deg
- Press Mode. Select 1: COMP — MODE 1
- Enter $\pi/4$ — ▭ SHIFT ×10ˣ (π) ▼ 4 ▶
- Enter radians "r" — SHIFT Ans (DRG▶) 2 =

ACTIVITIES

1. Two concentric circles have radii of 15.7 meters and 29.2 meters. Using a central angle of 84^0 , find the area of the sector between the two circles.

Solution:

Radian measure is used in finding the area of a sector of a circle. The area A is of a sector is proportional to its central angle θ. That is, if r is the radius of the circle then

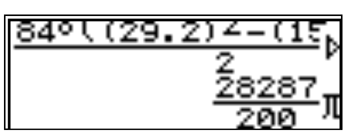
(area of sector A)/(area of circle) = (central angle of sector)/(central angle of circle)

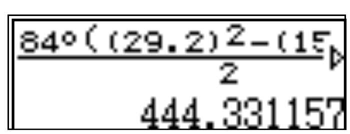
$$A/(\pi r^2) = \theta/2\pi$$

or equivalently,

$$A = (\theta r^2)/2$$

We solve for the difference of the two areas directly, that is, the area of the sector between the two circles is given by $\dfrac{84^0((29.2)^2-(15.7)^2)}{2}$.



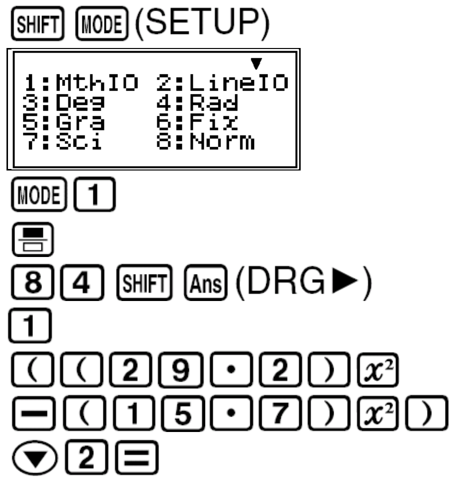


Thus, the area of the sector between the two circles is 444.331157 square meters.

Note that in the expression for the area, we assume that 84^0 is in radians. When we enter 84^0 in the calculator, we obtain the radian equivalence of 84 degrees.

[Operations]

- Press SHIFT SET UP
- Select 4: Rad
- Press Mode. Select 1: COMP
- Enter quotient
- Enter 84 degrees "o"
- Select 1
- Enter $((29.2)^2$
- Enter $-(15.7)^2$
- Enter 2



2. The end of a pendulum of length 40.2 cm travels an arc length of 5.6 cm as it swings through an angle θ. Find the measurement of the angle in

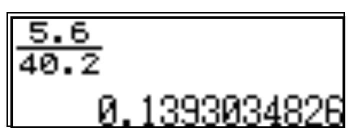
a. radians
b. degrees

Solution:

If r is the radius of a circle and θ is the radian measure of a central angle that intercepts on the circle an arc of length s then

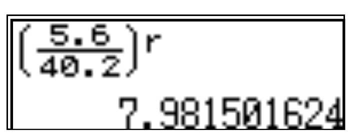
$$s = r\theta$$

The measurement of the angle can be computed using the formula $\theta = s/r$ where $s = 5.6$ and $r = 40.2$. That is $\theta = 5.6/40.2$. We have the following screen dumps from the calculator.



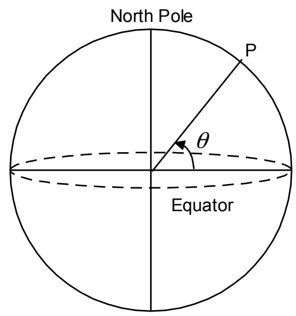
a. The answer is approximately 0.1393034829 radians
(Here, the calculator is set in radian mode)

b. The answer is approximately 7.98 degrees.



(Here the calculator is set in degree mode)

3. The latitude of a location point P, is the angle θ formed by a ray drawn from the center of the earth to the equator and from the center of the earth to P. Assume that the radius of the earth is about 4000 miles.

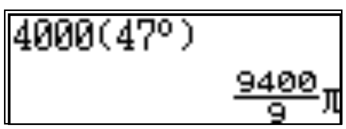


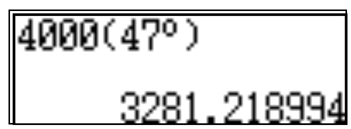
a. Seattle is 47^0 north latitude. How far north of the equator is Seattle?
b. Part of the US Canadian border is 49^0 north latitude. How far south of the border is Seattle?
c. Glasgow, Montana is due north of Albuquerque, Mexico. Find the distance between Glasgow (48 degrees, 13 minutes, $48^0\ 13^1$ north latitude) and Albuquerque (35 degrees, 7 minutes, $35^0\ 7^1$ north latitude).

Solution:

a. We use the formula $s = r\theta$, where we express 47^0 in radians. We have

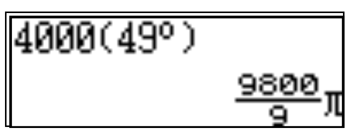
$$s = (4000 \text{ miles})\ (47^0) \approx 3281.218994.$$

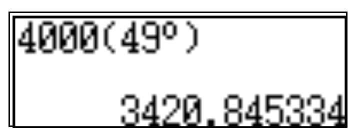




Thus, Seattle is approximately 3281 miles north of the equator.

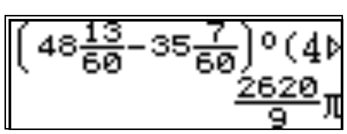
b. In the formula $s = r\theta$, we calculate $s =(4000 \text{ miles})\ (49^0) \approx 3420.845334$.

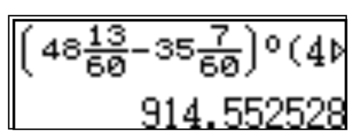




The US- Canadian border is about 3421 miles north of the equator. Since 3421-3281 is 140 miles, then Seattle is 140 miles south of the border.

c. We first get the radian equivalence of $48^0\ 13^1 - 35^0\ 7^1$, the measure of the central angle between the two cities. Then we enter this value as θ together with the radius $r = 4000$ in the formula $s = r\theta$. We obtain the ff:





The distance between the two cities is approximately 915 miles.

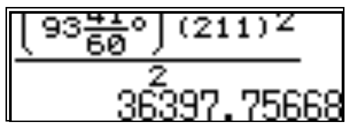
EXERCISES

1. Part of a baseball field and diamond is to be planted with grass seed. Determine the area of the field if the angle between the first and third baselines is $93^0\ 41^1$ and the radius of the sector to be seeded is 211 feet from home plate.

Solution:

The area of the sector is

$$A = (\theta r^2)/2$$
$$A = ((93^0\ 41^1 \text{ in radians})(211 \text{ feet})^2/2$$

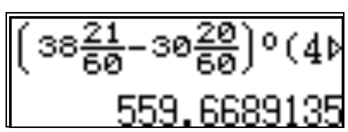


The area of the sector is approximately 36,398 square feet.

2. Charleston, West Virginia, is due north of Jacksonville, Florida. Find the distance between Charleston(latitude $38^0 21^1$), Jacksonville(latitude $30^0 20^l$). Assume that the radius of the earth is about 4000 miles.

Solution:

Following the solution in Example 3, item 3, we obtain approximately 560 miles.



Trigonometric Identities

A scientific calculator can be a very helpful tool in investigating the existence of an identity. The table feature of the calculator can provide the functional values of the left and right hand expressions, and can be a means to explore the validity of the given equality.

ACTIVITY

A function is even if $f(-x) = f(x)$ for all x in the domain of f. A function f is odd if $f(-x)= -f(x)$ for all x in the domain of f. Which of the trigonometric functions have the even property? the odd property?

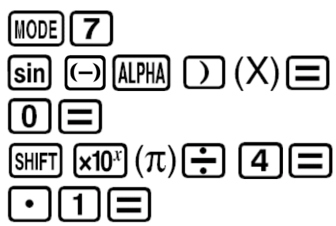
Solution:

Let us first illustrate the problem for the sine function. We verify whether the sine function is even, that is, we check if $\sin(-x) = \sin x$.

We use the following calculator keystrokes:

[Operations]

- Press Mode. Select 7: TABLE
- Enter $\sin(-x)$
- Enter start value 0
- Enter end value π /4
- Enter step value 0.1



Use the same process for $\sin(x)$.

Here are a few lines from the tables generated:

TABLE for $\sin(-x)$

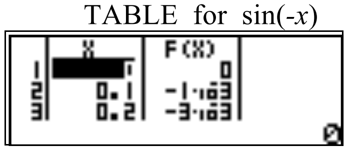
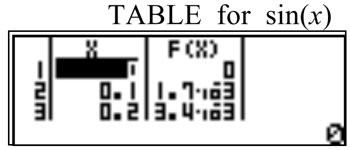


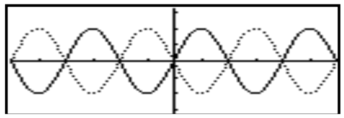
TABLE for $\sin(x)$



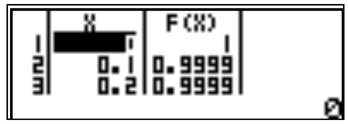
We obtain the conclusion from the partial list of data that $\sin(-x) \neq \sin(x)$.

Moreover, we note that $\sin(-x) = -\sin(x)$. The graphs of the functions are symmetric with respect to the x axis. Thus the sine function is an odd function.

The graphs of $\sin(x)$ (black) and sine $(-x)$ (dotted) show that indeed $\sin(-x) = -\sin(x)$. The screen dump that follows is an output from a graphics calculator. Without the graphics calculator, the student may construct the graphs by using values generated by the table feature of the scientific calculator as a guide.



The cosine function is an even function and not an odd function. Tables for cos(-*x*) and cos (*x*) will yield the same values as follows:



Note that in exploring the cosine function using the table feature we also use the same start, end values and increment of 0.1.

The tangent function is an odd function and not an even function as seen from the following output:

TABLE for tan(-*x*)

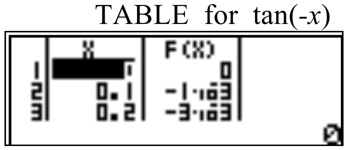
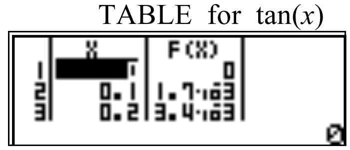


TABLE for tan(*x*)



For the tangent function, we use a start value of 0, end value of 1 and increment of 0.1.

From the fact that $\csc x = \dfrac{1}{\sin x}, \sec x = \dfrac{1}{\cos x}$ and $\cot x = \dfrac{1}{\tan x}$ and the result obtained above we conclude that

$$\csc(-x) = \frac{1}{\sin(-x)} = -\frac{1}{\sin x} = -\csc x$$

$$\sec(-x) = \frac{1}{\cos(-x)} = \frac{1}{\cos x} = \sec x$$

$$\cot(-x) = \frac{1}{\tan(-x)} = -\frac{1}{\tan x} = -\cot x$$

These results imply that **cosine and secant are even functions and cosecant is an odd function.**

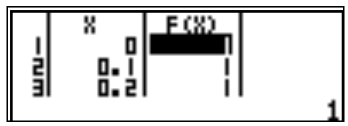
EXERCISES

1. Simplify $\sin^2(-x) + \cos^2(-x)$.

Solution:

Using the table feature, we enter the expression $\sin^2(-x) + \cos^2(-x)$. We use the same start, end values and increment given earlier while exploring the sine and cosine functions.

As we scroll down the data, we can verify that the value of $\sin^2(-x) + \cos^2(-x)=1$.



We can establish the identity $\sin^2(-x) + \cos^2(-x)=1$ from the results we obtained in the exploratory activity.

Note that $\sin^2(-x) + \cos^2(-x) = [\sin(-x)]^2 + [\cos(-x)]^2$
$= (-\sin x)^2 + (\cos x)^2$
$= 1$

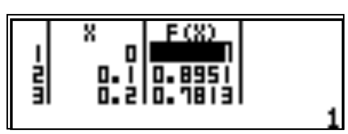
In fact, the identity $\sin^2(-x) + \cos^2(-x)=1$ or equivalently, the identity $\sin^2 x + \cos^2 x = 1$ is called the **Pythagorean identity.**

2. Is the following expression an identity?

$$\frac{\sin^2(-x) - \cos^2(-x)}{\sin(-x) - \cos(-x)} = \cos x - \sin x$$

Solution:

The expression $\dfrac{\sin^2(-x) - \cos^2(-x)}{\sin(-x) - \cos(-x)} = \cos x - \sin x$ is an identity. This can be verified by entering the left hand expression and right hand expression in the table feature of the calculator. We get the following table of values:



3. Calculus

Epsilon-Delta Limit Investigations

One of the more difficult topics in calculus is the $\varepsilon - \delta$ definition of the limit of a given function.

Suppose f is a function defined at every number in some open interval containing a except possibly at the number a itself. The limit of $f(x)$ as x approaches a is L, written as

$$\lim_{x \to a} f(x) = L$$

if the following statement is true:
Given any $\varepsilon > 0$, however small, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

We start the chapter by providing explorations on the epsilon-delta limit definitions with the aid of the scientific calculator.

ACTIVITY

Consider $f(x) = x^3 + 5x^2 + 10x + 98$. Investigate $\lim_{x \to 3} f(x) = 200$ by finding a $\delta > 0$ given the following values for $\varepsilon = 1, 0.1, 0.01, 0.001$ such that

$$\text{if } 0 < |x - 3| < \delta \text{ then } |(x^3 + 5x^2 + 10x + 98) - 200| < \varepsilon .$$

Solution:

We solve for $\delta > 0$ given each ε as follows:

First, we solve numerically the following equations

$$x^3 + 5x^2 + 10x + 98 = 200 - \varepsilon ,\ x^3 + 5x^2 + 10x + 98 = 200 + \varepsilon$$

or equivalently,

$$x^3 + 5x^2 + 10x - 102 + \varepsilon = 0,\ x^3 + 5x^2 + 10x - 102 - \varepsilon = 0$$

for the respective solutions x_1 and x_2 near 3.

Consider $\varepsilon = 1$. We get the equations

$$x^3 + 5x^2 + 10x - 101 = 0 \text{ and } x^3 + 5x^2 + 10x - 103 = 0.$$

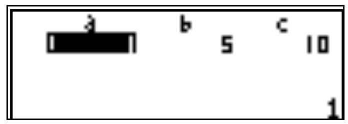
We use the calculator to solve the equations.

To illustrate, we solve the equation $x^3 + 5x^2 + 10x - 101 = 0$ for x_1. We have the following calculator keystrokes:

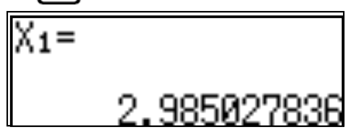
[Operations]

- Press Mode. Select 5: EQN [MODE] [5]
- Select cubic equation [4]
- Enter coefficients of the equation in the table then press [=].

Example: Enter 1,5,10,-101 [1] [=] [5] [=] [1] [0] [=] [(-)] [1] [0] [1]

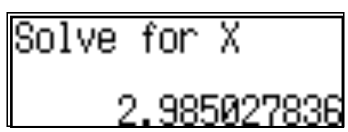


- Enter equal sign to get root [=]



Another approach:

- Press Mode. Select 1:COMP [MODE] [1]
- Enter $x^3 + 5x^2 + 10x + 98 = 200 - 1$ directly
- Enter $x^3 + 5x^2$ [ALPHA] [)] (X) [SHIFT] [x^2] (x^3) [+] [5] [ALPHA] [)] (X) [x^2]
- Enter $10x + 98$ [+] [1] [0] [ALPHA] [)] (X) [+] [9] [8]
- Enter $= 200 - 1$ [ALPHA] [CALC] (=) [2] [0] [0] [−] [1]
- Solve [SHIFT] [CALC] (SOLVE)



We obtain $x_1 = 2.98503$.

We repeat either of the two methods given above to solve the equation $x^3 + 5x^2 + 10x - 103 = 0$ for x_2.

We get the answer $x_2 = 3.01488$.

Next, we compute for the indicated distances $\delta_1 = a - x_1$ and $\delta_2 = a - x_2$:

We have $\delta_1 = 3 - x_1 = 3 - 2.98503 = 0.01497$.

Also, $\delta_2 = x_2 - 3 = 3.01488 - 3 = 0.01488$

Then we choose $\delta = \min(\delta_1, \delta_2) = 0.01488$.

We use the same procedure for $\varepsilon = 0.1, 0.01, 0.001$

The table that follows gives the values for x_1, x_2, δ for each of the ε.

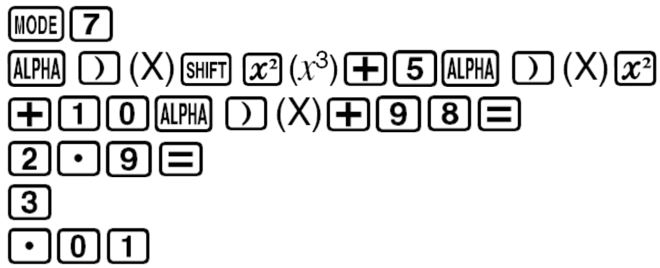
ε	x_1	x_2	δ_1	δ_2	δ =min(δ_1 , δ_2)
1.0	2.98503	3.01488	0.01497	0.01488	0.01488
0.1	2.99851	3.00149	0.00149	0.00149	0.00149
0.01	2.99985	3.00015	0.00015	0.00015	0.00015
0.001	2.99999	3.00001	0.00001	0.00001	0.00001

We can investigate further the fact that $\lim_{x\to 3} f(x) = 200$ using the tabular feature of the calculator. We generate tables of values near $x = 3$.

First, we observe how $f(x)$ behaves as x approaches 3 from values to the left of 3.

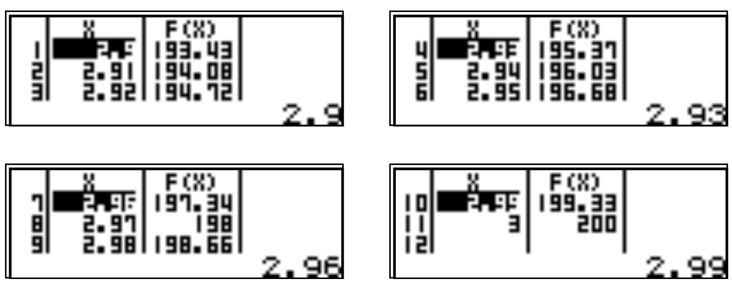
[Operations]

- Press Mode. Select 7: TABLE
- Enter $x^3 + 5x^2$
- Enter $+10x + 98$
- Enter start value 2.9
- Enter end value 3
- Enter step value 0.01



(any start, end or step values may be experimented on to observe the limit as x approaches 3 from the left)

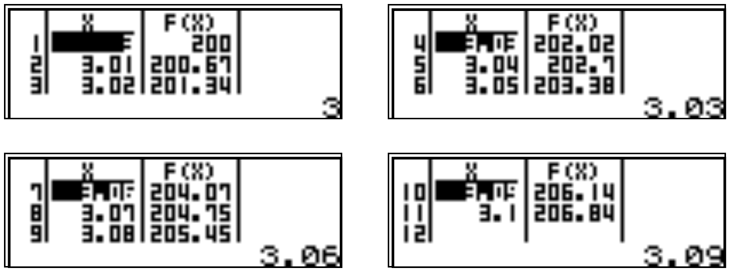
The following table of values is generated:



As seen from the table, when x approaches 3 from the left, $f(x)$ approaches 200.

We generate another table of values this time we let x approach 3 from the right. This time we use start value 3, end value 3.1 and step 0.01.

We obtain the following table of values:



As seen from the table (starting from $x = 3.1$ to $x = 3$), when x approaches 3 from the right, $f(x)$ approaches 200.

EXERCISES

1. Determine a $\delta > 0$ such that if $0 < |x+3| < \delta$ then $|(2x^2+5x+3)-6| < 0.6$.

Solution:

We solve numerically the following equations

$$2x^2+5x+3 = 6+0.6,\;\; 2x^2+5x+3 = 6-0.6$$

for the respective solutions x_1 and x_2 near -3.

We obtain $x_1 = -3.0837, x_2 = -2.9121$

Then we have δ_1= -3+3.0837= 0.0837 and δ_2= -3 +2.9121=0.0879.

We choose $\delta = \min(\delta_1, \delta_2) = 0.0837$

2. The circular top of a coffee table is to have an area differing from 225π square inches by less than 4 square inches. How accurately must the radius of the top be measured?

Solution:

Let r inches be the radius of the circular top and $A(r) = \pi r^2$

The area is 225π square inches when the radius is 15 inches. We wish to determine how close r must be to 15 in order for $A(r)$ to be within 4 of 225π. That is, if ε =4, we wish to find a δ >0 such that

$$\text{if } 0 < |r-15| < \delta \text{ then } |A(r) - 225\pi| < 4$$

Equivalently, we have

$$\text{if } 0 < |r-15| < \delta \text{ then } |\pi r^2 - 225\pi| < 4$$

We solve numerically the following equations

$$\pi r^2 = 225\pi - 4,\; \pi r^2 = 225\pi + 4$$

for the respective solutions r_1 and r_2 near 15.

Using the calculator, we obtain r_1=14.9575 and r_2=15.0424

Thus, the radius of the top should be approximately between 14.9575 and 15.0424 inches in order for the circular top to have an area differing from 225π square inches by less than 4 square inches.

Limits

We write

$$\lim_{x \to c} f(x) = L \text{ or } f(x) \to L \text{ as } x \to c$$

if the functional value $f(x)$ is close to the single real number L whenever x is close to, but not equal to, c (on either side of c).

In this section, we illustrate how to study limits using a numerical approach – in particular using the tabular feature of the calculator.

ACTIVITIES

The following activities illustrate that the existence of a limit at c has nothing to do with the value of the function at c.

Estimate the following limits:

1. $\lim_{x \to 2} \dfrac{x^3 - 8}{x - 2}$
2. $\lim_{x \to 0} \dfrac{e^{2x} - 1}{x}$

Solution:

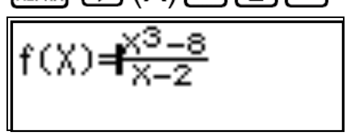
1. We construct a table of values with x approaching 2. First, we let x approach 2 from the left(from values less than 2). Next, we let x approach 2 from the right (from values greater than 2). We set up a table of values with x approaching 2 from the left using the following calculator keystrokes:

[Operations]

- Press Mode. Select 7: TABLE [MODE] [7]

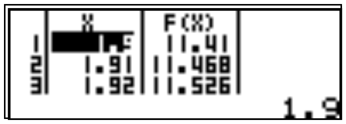
Enter $\dfrac{x^3 - 8}{x - 2}$ in $f(x)$ as follows:

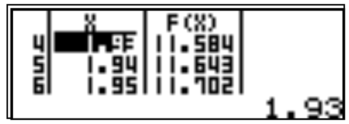
- Access quotient sign [▭/▭]
- Enter $x^3 - 8$ [ALPHA] [)] (X) [SHIFT] [x^2] (x^3) [−] [8] [▼]
- Enter $x - 2$ [ALPHA] [)] (X) [−] [2] [=]

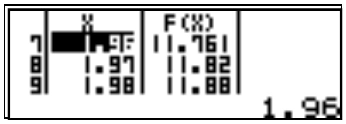


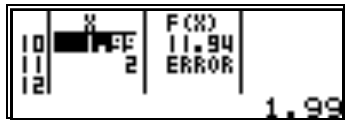
- Enter start value 1.9 [1] [•] [9] [=]
- Enter end value 2 [2] [=]
- Enter step value 0.01 [•] [0] [1] [=]

We obtain the following table output from the calculator:









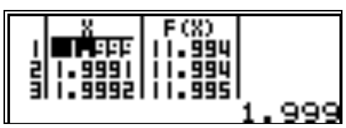
Observe that we cannot simply substitute $x = 2$ because the function is not defined at $x = 2$. In fact, the calculator gives us an error message at $x = 2$. However, as x approaches 2 from the left, $f(x)$ approaches 12.

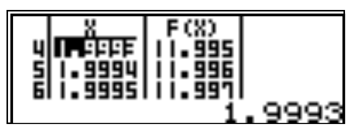
To confirm further that the limiting value of the function is 12 as x approaches 2 from the left, we can generate a table of values again, this time selecting a closer range of values to 2.(Note that one can experiment with different x values to explore the behavior of $f(x)$ for x near 2 but not equal to 2.)

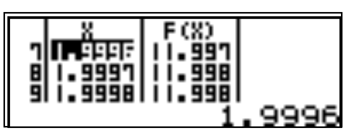
[Operations]

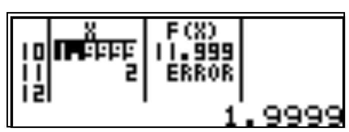
- Enter start value 1.999 — 1 . 9 9 9 =
- Enter end value 2 — 2 =
- Enter step value 0.0001 — . 0 0 0 1 =

We obtain the following table of values:









As we can see from the table, we get a clearer picture that as x approaches 2 from the left, $f(x)$ approaches 12.

We now set up table of values with x approaching 2 from the right. We use the following start, end and step values: for Table A: start value: 2, end value: 2.1, step value 0.01; for Table B: start value: 2, end value 2.001, step value 0.0001.

We obtain the following tables.

Table A:

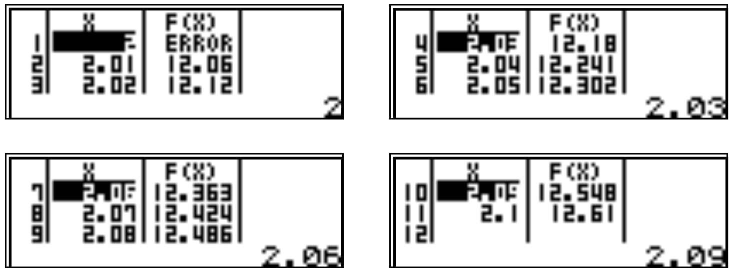
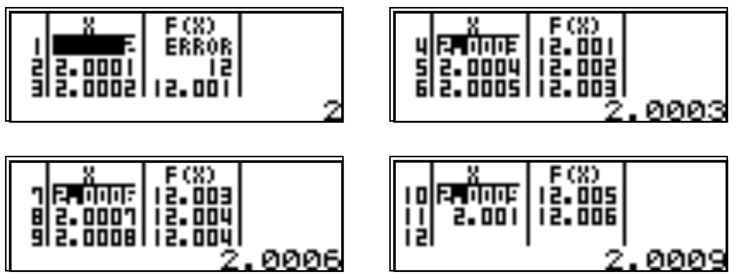


Table B:



For Table A, we start with $x = 2.1$. At $x = 2.1$, $f(x) = 12.61$, and slowly approaches 12, as x approaches 2 from values greater than 2. For Table B, we start with $x = 2.001$. At $x = 2.001$, $f(x) = 12.006$, and slowly approaches 12, as x approaches 2 from values greater than 2. We notice that as x approaches 2 from either side, $f(x)$ approaches 12. This suggests that the limit is 12. We write $\lim_{x \to 2} \frac{x^3 - 8}{x - 2} = 12$.

2. The function $f(x) = \frac{e^{2x} - 1}{x}$ is not defined at $x = 0$. We construct tables with x approaching 0 from both sides.

We first approach 0 from the left. We generate Table A using start value: -0.001 , end value: 0, step value: 0.0001 . We next approach 0 from the right. We generate Table B using start value: 0, end value: 0.001, step value: 0.0001 . Note that we read Table B starting with *x*= 0.001 with *f*(*x*)= 2.002 approaching 0 from values greater than 0.

Table A and Table B are given as follows:

Table A:

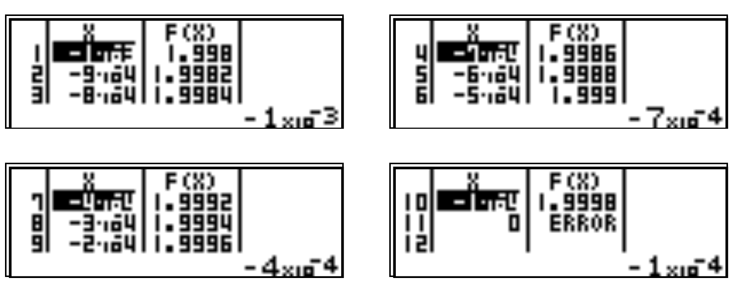
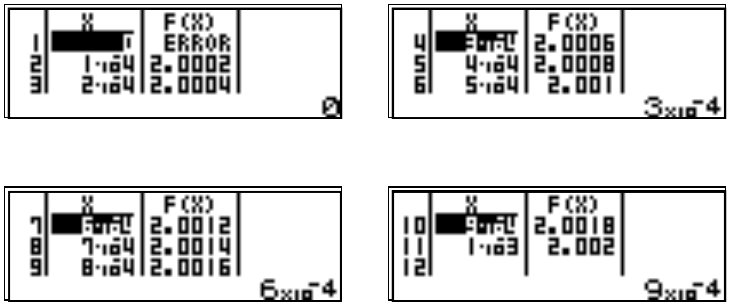


Table B:



The tables suggest that $\lim_{x\to 0}\frac{e^{2x}-1}{x}$ = 2.

EXERCISES

1. Estimate $\lim_{x\to o}(1+x)^{\frac{1}{x}}$.

Solution:

The function $f(x) = (1+x)^{\frac{1}{x}}$ is not defined at $x = 0$. We generate tables of values. We use same start, end and step values as in the previous activity b).

We obtain the following tables:

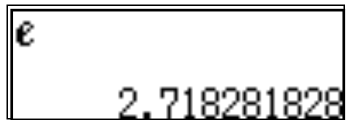
x	*f*(*x*)		*x*	*f*(*x*)
-0.001	2.7196		0	error
-0.0009	2.7195		0.0001	2.7181
-0.0008	2.7194		0.0002	2.7180
-0.0007	2.7192		0.0003	2.7179
-0.0006	2.7191		0.0004	2.7177
-0.0005	2.719		0.0005	2.7176
-0.0004	2.7188		0.0006	2.7175
-0.0003	2.7187		0.0007	2.7173
0.0002	2.7186		0.0008	2.7172
-0.0001	2.7184		0.0009	2.7171
0	error		0.001	2.7169

The tables suggest that $\lim_{x\to o}(1+x)^{\frac{1}{x}} = 2.718$. It can be shown that as you take *x* values closer and closer to zero the value of this limit is non-other than the Euler constant e with value approximately 2.718281828.

We can verify the value of the Euler constant e using the following calculator commands:

[Operations]

- Press Mode. Select 1: COMP — [MODE] [1]
- Enter e — [ALPHA] [×10^x] (e) [=]



2. Let $f(x) = \frac{|x|}{x}$. Does $\lim_{x \to o} f(x)$ exist?

Solution:

We generate tables of values. We use same start, end and step values as in the previous activity b).

x	***f*(*x*)**		***x***	***f*(*x*)**
-0.001	-1		0	error
-0.0009	-1		0.0001	1
-0.0008	-1		0.0002	1
-0.0007	-1		0.0003	1
-0.0006	-1		0.0004	1
-0.0005	-1		0.0005	1
-0.0004	-1		0.0006	1
-0.0003	-1		0.0007	1
0.0002	-1		0.0008	1
-0.0001	-1		0.0009	1
0	error		0.001	1

The tables show that $f(x)$ does not approach the same limit as x approaches 0 from both sides. The limit as we approach zero from the left is –1. The limit as we approach zero from the right is 1. Thus, $\lim_{x \to o} f(x)$ does not exist.

Derivative at a point

The derivative of the function $f(x)$ with respect to x is the function $f'(x)$ given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

In the activities that follow we show how the calculator may be used to approximate the derivative using the limiting value of the slope of the secant lines by varying the values for h.

ACTIVITIES

Find the slope of the line that is tangent to the graph of the function $f(x) = \sqrt{x^2 + 2x - \sqrt{3x}}$ at the point where $x = 3.85$.

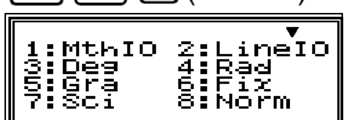
1. Approximate the slope of the tangent line using the limit of the difference quotient $\frac{f(x+h) - f(x)}{h}$ by taking the following values of h = -0.03,-0.02, -0.01, 0.01, 0.02, 0.03 and calculating up to five decimal places.
2. Compare your answer in a) by determining $f'(385)$ directly from the calculator.

Solution:

1. We first determine the value of $f(x) = \sqrt{x^2 + 2x - \sqrt{3x}}$ at $x = 3.85$ using the calculator.

[Operations]

• Press SHIFT SET UP	SHIFT MODE (SETUP)
• Select 2:Line I0	SHIFT MODE 2 (LineIO)



• Press Mode. Select 1:COMP	MODE 1
• Access square root sign	√■
• Enter 3.85^2+2(3.85) –	3 • 8 5 x^2 + 2 (3 • 8 5) −
• Access square root sign	√■
• Enter 3(3.85)	3 (3 • 8 5) =

We obtain $f(3.85) = \sqrt{(3.85)^2 + 2(3.85) - \sqrt{3(3.85)}} = 4.3731$

We now generate the values of the difference quotient $\frac{f(x+h) - f(x)}{h} = \frac{f(3.85+h) - f(3.85)}{h} = \frac{f(3.85+h) - 4.3731}{h}$ using the table feature of the calculator.

We use the following commands:

[Operations]

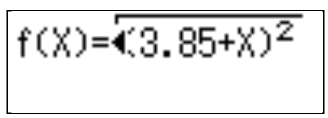
- Press SHIFT SET UP — [SHIFT] [MODE] (SETUP)
- Select 1:MthI0 — [SHIFT] [MODE] [1] (MthIO)

- Press Mode. Select 7:TABLE — [MODE] [7]

For the value of the function in the table, we enter enter "*x*" instead of "*h*" in the quotient $\dfrac{f(3.85+h)-4.3731}{h}$.

That is, we enter as $f(x)=\dfrac{1}{x}(\sqrt{(3.85+x)^2}+2(3.85+x)-\sqrt{3(3.85+x)}-4.3731)$

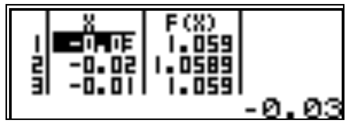
- Access quotient sign — [▭/▭]
- Enter $\dfrac{1}{x}($ — [1] [▼] [ALPHA] [)] (X) [▶] [(]
- Access square root sign — [√▭]
- Enter $(3.85+x)^2$ — [(] [3] [•] [8] [5] [+] [ALPHA] [)] (X) [)] [x^2]



- Enter $+2(3.85+x)$ — [+] [2] [(] [3] [•] [8] [5] [+] [ALPHA] [)] (X) [)]
- Enter $-\sqrt{3(3.85+x)}$ — [−] [√▭] [3] [(] [3] [•] [8] [5] [+] [ALPHA] [)] (X) [)] [▶] [▶]
- Enter $-4.371)$ — [−] [4] [•] [3] [7] [3] [1] [)] [=]
- Enter start value, -0.03 — [(−)] [•] [0] [3] [=]
- Enter last value, 0.03 — [•] [0] [3] [=]
- Enter step value, 0.01 — [•] [0] [1] [=]

We get the following table of values:

h	-0.03	-0.02	-0.01	0	0.01	0.02	0.03
$\dfrac{f(x+h)-f(x)}{h}$	1.059	1.059	1.059	undefined	1.058	1.058	1.058



We observe that the $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ is about 1.06.

Note that when h =0, the calculator gives an error message since the difference quotient becomes undefined.

2. To calculate $f'(385)$ directly from the calculator, we use the following commands:

[Operations]

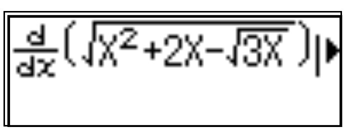
- Press SHIFT SET UP — SHIFT MODE (SETUP)
- Select 1:MthI0 — SHIFT MODE 1 (MthIO)

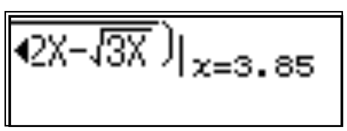
- Press Mode. Select 1:COMP — MODE 1
- Select derivative — SHIFT ∫ ($\frac{d}{dx}$■)

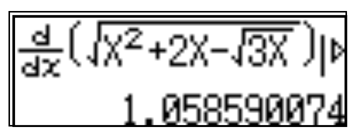
Enter $\sqrt{x^2 + 2x - \sqrt{3x}}$

- Access square root sign — √■
- Enter $x^2 + 2x$ — ALPHA) (X) x^2 + 2 ALPHA) (X)
- Enter $-\sqrt{3x}$ — − √■ 3 ALPHA) (X) ▶ ▶ ▶
- Enter 3.85 — 3 • 8 5 =

We obtain $f'(385) = 1.0585 \approx 1.059$; which is very close to the value of the limit obtained in a).







EXERCISES

1. Consider $f(x) = 2^x$.

a. Estimate the value of $f'(0)$ using the limit of the difference quotient by taking successively smaller values of h = 0.1, 0.01, 0.001, 0.0001, -0.1,-0.01, -0.001,-0.0001.

b, Compare your answer in a) by determining $f'(0)$ directly from the calculator.

Solution:

a. We use the table feature of the calculator to obtain the values of the difference quotient $\frac{f(0+h)-f(0)}{h} = \frac{2^h-1}{h}$ given the different values of *h*.

[Operations]

- Press SHIFT SET UP — [SHIFT] [MODE] (SETUP)
- Select 1:MthI0 — [SHIFT] [MODE] [1] (MthIO)

- Press Mode. Select 7:TABLE — [MODE] [7]

That is, we enter as $f(x) = \frac{2^x-1}{x}$ (we use "*x*" instead of "*h*")

- Access quotient sign — [fraction key]
- Enter $\frac{1}{x}($ — [1] [▼] [ALPHA] [)] (X) [▶] [(]
- Enter $2^x - 1)$ — [2] [$x^{\blacksquare}$] [ALPHA] [)] (X) [▶] [−] [1] [)] [=]

Note that we have to determine the value of $f(x) = \frac{2^x-1}{x}$ at x = 0.1, 0.01, 0.001, 0.0001, -0.1,-0.01, -0.001,-0.0001. Unlike the preceding example, the difference or step value between the *x* values are not the same. In this case, we cannot generate one table of values. We have to calculate the value of $f(x) = \frac{2^x-1}{x}$ one at a time for each *x* value.

Consider for example, *x* = 0.1, we enter *x* = 0.1 as start value and enter *x* = 0.2 as end value with step 0.1. Note that you can use any end value and step value, as long as the answer at *x* = 0.1 can be determined. The table feature of the calculator may be used in this manner, in cases when different functional values need be determined for without a common step value, to speed up the process.

- Enter start value, 0.1 — [•] [1] [=]
- Enter last value, 0.2 — [•] [2] [=]
- Enter step value, 0.1 — [•] [0] [1] [=]

We repeat the process to find the difference quotient at the next value*, x* = 0.01, etc.
We get the following results:

h	0.1	0.01	0.001	0.0001	-0.1	-0.01	-0.001	-0.0001
$\frac{2^h-1}{h}$	0.718	0.696	0.693	0.693	0.67	0.691	0.693	0.693

Thus, $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ = 0.693.

b. Calculate $f'(0)$ directly from the calculator, we also obtain 0.693.

[Operations]

• Press SHIFT SET UP	SHIFT MODE (SETUP)
• Select 1:MthI0	SHIFT MODE 1 (MthIO)

`

• Press Mode. Select 1:COMP	MODE 1
• Select derivative	SHIFT $\int$ ($\frac{d}{dx}$▮)
• Enter 2^x	2 $x^{■}$ ALPHA) (X) ▶ ▶
• Enter 0	0 =

2. Repeat exercise 1 to find $f'(6)$ given $f(x) = \dfrac{8}{x-2}$. Select varying values for *h*.

Answer: $f'(6)$=-$\dfrac{1}{2}$

The interpretation of the derivative as the slope of a tangent line gives us information about how a function and its graph behave.

We now show how derivatives are used to study the graph of a function.

ACTIVITIES

Consider the function

$$f(x) = 0.25x^4 - 2x^3 + 1.5x^2 + 21x + 15$$

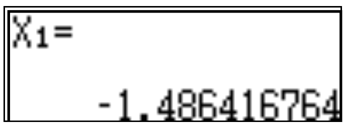
1. Find the critical values, the interval/s where $f(x)$ is increasing; the interval/s where $f(x)$ is decreasing.
2. Approximate the $x-$coordinates of the inflection points of f. Find the intervals where the graph of f is concave up and concave down.
3. Find the local extrema for the function.

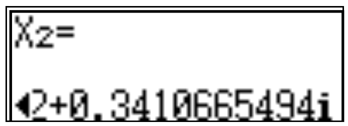
Solution:

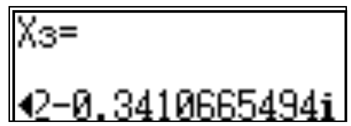
1.The derivative of the given function is $f'(x) = x^3 - 6x^2 + 3x + 21$. The critical values are values of x in the domain of f where $f'(x) = 0$ or $f'(x)$ does not exist. Since $f'(x) = x^3 - 6x^2 + 3x + 21$ exists for all values of x in the domain of f, we solve $f'(x) = 0$.

[Operations]

- Press Mode. Select 5: EQN — MODE 5
- Select 4: $ax^3+bx^2+cx+d=0$ — 4
- Enter a value, 1 — 1 =
- Enter b value,-6 — (−) 6 =
- Enter c value,3 — 3 =
- Enter d value, 21 — 2 1 =







We obtain the only critical value $x = -1.49$. The other two roots of $f'(x)$ are complex numbers

Now, to determine the interval/s where $f(x)$ is increasing and decreasing, we use the following result for a continuous function g on the closed interval $[a,b]$: if $g'(x) > 0$ for all x in (a,b), then g is increasing on (a,b) and if $g'(x) < 0$ for all x in (a,b), then g is decreasing on (a,b).

We check the signs of $f'(x)$ before and after the critical value $x = -1.49$. We use the table feature of the calculator.

[Operations]

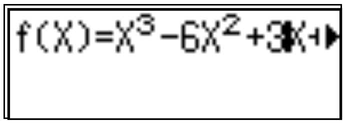
- Press Mode. Select 7: TABLE — MODE 7

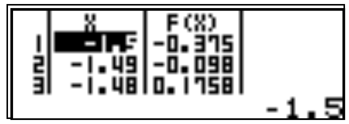
  Enter the given function $f'(x) = x^3 - 6x^2 + 3x + 21$
- Enter x^3 — ALPHA) (X) SHIFT x^2 (x^3)
- Enter $-6x^2$ — − 6 ALPHA) (X) x^2
- Enter $+3x+21$ — 3 ALPHA) (X) + 2 1

Enter as the start value any number less than -1.49 and as the end value any number greater than -1.49. This will do, since we just want to know the sign of the derivative before and after the critical value. For instance , we can have start value -1.5 and end value -1.48. A reasonable step or difference value will be 0.01.

- Enter start value, -1.5 — (−) 1 • 5 =
- Enter end value -1.48 — (−) 1 • 4 8 =
- Enter step value, 0.01 — • 0 1 =

We obtain the following table:



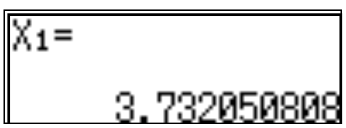


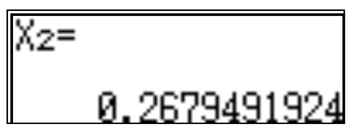
The sign of the first derivative is positive at -1.48 (after -1.49) and negative at -1.5 (before -1.49). This means f is increasing on $(-1.49, \infty)$ and decreasing on $(-\infty, -1.49)$.

2. The second derivative of the given function is $f''(x) = 3x^2 - 12x + 3$. In general, an inflection point is a point on the graph of the function where concavity changes. To determine the x-coordinates of the possible inflection points of f, we solve $f''(x) = 3x^2 - 12x + 3 = 0$.

[Operations]

- Press Mode. Select 5: EQN [MODE] [5]
- Select 3: $ax^2+bx+c=0$ [3]
- Enter a value, 3 [3] [=]
- Enter b value, -12 [(-)] [1] [2] [=]
- Enter c value, 3 [3] [=]





We obtain the values $x_1 = 3.73$ and $x_2 = 0.27$

Now, we determine the intervals where the graph of f is concave up and concave down.

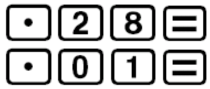
We use the following result for a function g that is differentiable in some open interval containing c: If $g''(c) > 0$, the graph of f is concave upward at $(c, g(c))$; if $g''(c) < 0$, the graph of g is concave downward at $(c, g(c))$.

We use the tabular feature of the calculator and determine the sign of the second derivative in the intervals before and after the possible inflection points.

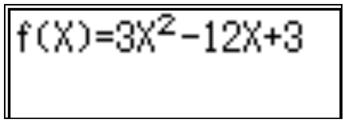
[Operations]

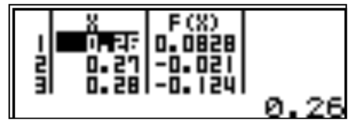
- Press Mode. Select 7: TABLE [MODE] [7]
  Enter the given function $f''(x) = 3x^2 - 12x + 3 = 0$
- Enter $3x^2$ [3] [ALPHA] [)] (X) [x^2]
- Enter $-12x + 3$ [−] [1] [2] [ALPHA] [)] (X) [+] [3]
  Enter as the start value any number less than 0.27 and as the end value any number greater than 0.27. For instance , we can have start value 0.26 and end value 0.28. A reasonable step or difference value will be 0.01.
- Enter start value, 0.26 [•] [2] [6] [=]

- Enter end value 0.28
- Enter step value, 0.01



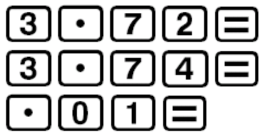
We obtain the following table:



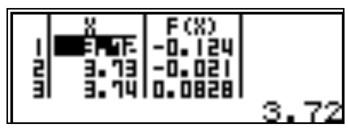


Enter as the start value any number less than 3.73 and as the end value any number greater than 3.73. For instance , we can have start value 3.72 and end value 3.74. A reasonable step or difference value will be 0.01.

- Enter start value, 3.72
- Enter end value 3.74
- Enter step value, 0.01

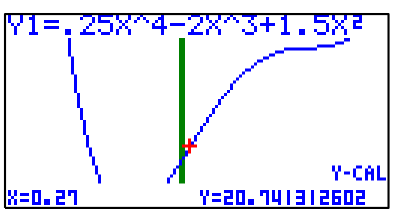


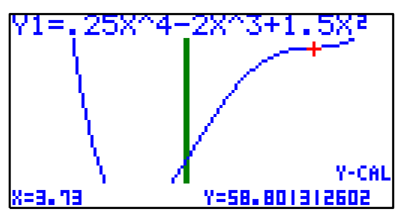
We obtain the following table:



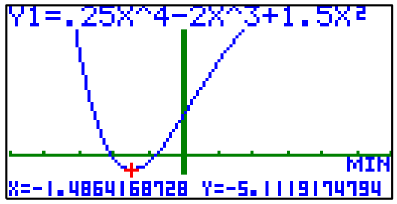
Note that the sign of the second derivative is positive at 0.26 (before 0.27) and negative at 0.28(after 0.27).
Moreover, the sign of the second derivative is positive at 3.74 (after 3.73) and negative at 3.72 (before 3.73).
This implies the graph of f is concave upward on $(-\infty, 0.27)$ and $(3.73, \infty)$. It is concave downward on $(0.27, 3.73)$. Inflection points are at $x_1 = 3.73$ and $x_2 = 0.27$.

The graph of the function given in the images below shows a change of concavity before and after the inflection points.





3. Since the function is f is increasing on $(-1.49, \infty)$ and decreasing on $(-\infty, -1.49)$, then there is a local minimum point at $x = -1.49$.



We will now give a derivative application to a real-world problem.

Combined SAT scores in the United States in the years 1967-1991 could be approximated by

$$T(t) = -0.01085t^3 + 0.5804t^2 - 10.12t + 962.4 \text{ on } [0,22]$$

where t is the number of years since 1967, and T is the combined SAT score average.

1. Based on this model, when (to the nearest year) was the average SAT score decreasing most rapidly?
2. When was it increasing most rapidly?

Solution:

The first derivative of T is given by

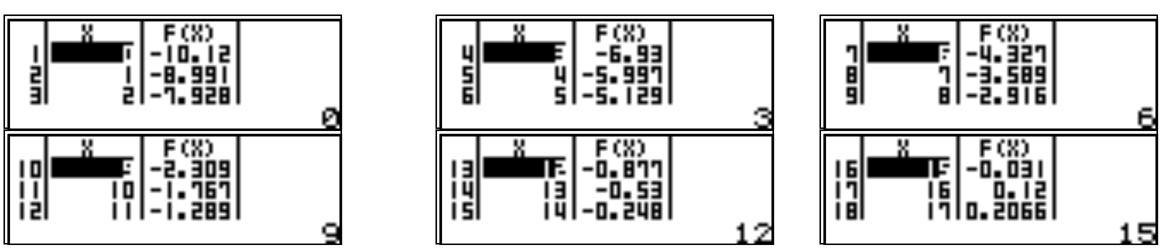
$$T'(t) = (-0.01085)3t^2 + (0.5804)2t - 10.12 = -0.03255t^2 + 1.1608t - 10.12.$$

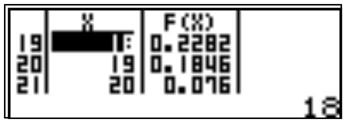
We analyze the signs of the first derivative to check where the function is decreasing and increasing most rapidly.

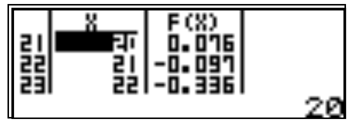
We use the tabular feature of the scientific calculator to answer the problem.

[Operations]

- Press Mode. Select 7: TABLE [MODE] [7]
  Enter the given function $-0.03255x^2 + 1.1608x - 10.12$
- Enter $-0.03255x^2$ [(-)] [•] [0] [3] [2] [5] [5] [ALPHA] [)] (X) [x^2]
- Enter $+1.1608x$ [+] [1] [•] [1] [6] [0] [8] [ALPHA] [)] (X)
- Enter -10.12 [−] [1] [0] [•] [1] [2] [=]
- Enter start value, 0 [0] [=]
- Enter end value,22 [2] [2] [=]
- Enter step value,1 [1] [=]







In the table generated, we can verify that the first derivative attains the highest (positive) value at x =18; f ’(18) = 0.22. Also, the lowest (negative) value was attained at x = 0, that is, f ’(0) = - 10.12. Thus, the average SAT score was increasing most rapidly in 1985 and decreasing most rapidly in 1967.

Applications of Derivatives to Economics

In economics, the cost, revenue and profit functions are related by the equation: Profit= Revenue – Cost. In this section, we study these economic quantities using algebraic and calculus concepts.

Activity 1. The table below contains price-demand and total cost data for the production of radial arm saws, where p is the wholesale price (in dollars) of a saw for an annual demand of x saws and C is the total cost (in dollars) of producing x saws.

x	p	C
950	240	130,000
1,200	210	150,000
1,800	160	180,000
2,050	120	190,000

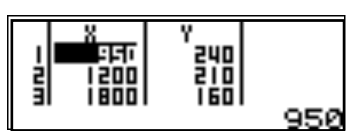
a. Find the wholesale price of a saw corresponding to an annual demand of 1,400 saws.
b. How many saws were produced for a total cost of $250,000?
c. What is the maximum profit?

Solution:

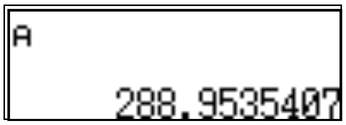
a. Our first step is to find a quadratic function $p(x)$ that will model the price demand data using x as the independent variable, and p as the dependent variable. We use a process called quadratic regression to find the function.

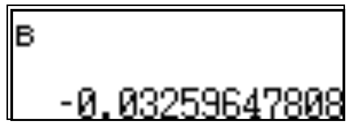
[Operations]

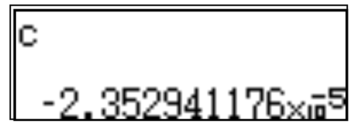
- Press Mode. Select 3: STAT [MODE] [3]
- Select 3: Quadratic Regression _+CX^2 [3]
- Enter each of the data in the table- x values in first column, p values in the second column. After all of the data is entered, press [AC].
-



- Calculate the regression curve A +BX + CX^2
  Calculate the constant term A SHIFT 1 5 1 =
  Calculate the coefficient of X, B SHIFT 1 5 2 =
  Calculate the coefficient of X^2, C SHIFT 1 5 3 =
- Whenever we want to refer to the same data again, SHIFT 1 2
  The data will be kept in the calculator's memory.







We obtain $A = 288.95354$, $B = -0.0325965$ and $C = -0.00002353$.

The quadratic function representing the price-demand function is given by

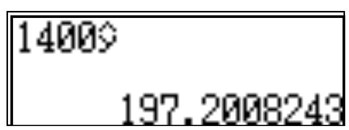
$$p(x) = -0.00002353x^2 - 0.0325965x + 288.95354.$$

This function models the price-demand data. This function is helpful in obtaining additional information; such as approximating price given the demand of the product, or determining the demand when price is given.

For instance, we can calculate the price corresponding to an annual demand of 1,400 saws as follows:

[Operations]

- Enter 1400 1 4 0 0
- Calculate the dependent variable $\hat{y}$ SHIFT 1 5 6 =

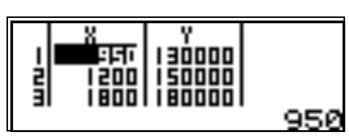


We get an answer of 197.20. This means the wholesale price of a saw corresponding to an annual demand of 1,400 saws is $ 197.20.

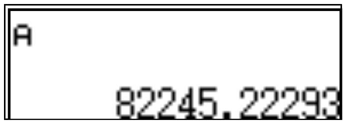
b. Now, using linear regression, we derive a linear function, that will model the cost data.

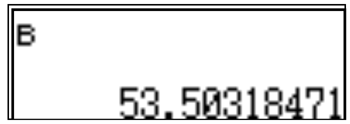
[Operations]

- Press Mode.Select 3:STAT MODE 3
- Select 2: Linear Regression A+BX 2
- Enter each of the data in the table- x values in first column, C values in the second column. After all of the data is entered, press AC.



- Calculate the regression line A +BX
  Calculate the constant term A — SHIFT 1 5 1 =
  Calculate the coefficient of X, B — SHIFT 1 5 2 =



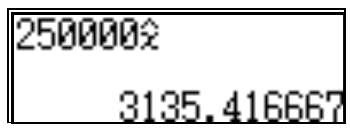


We obtain $A = 82245.2229$ and $B = 53.50318471$. The linear function that models the cost function is $C(x) = 53.5031847x + 82245.2229$.

To determine the number of saws that are produced for a total cost of $250,000, we use the following calculator commands:

[Operations]

- Enter 250000 — 2 5 0 0 0 0
- Calculate the independent variable $\hat{x}$ — SHIFT 1 5 4 =



The answer we get is 3135.41. This means that the approximate number of saws that are produced for a total cost of $ 250,000 is approximately 3135.

c.. From a), the price demand function is $p(x) = -0.00002353x^2 - 0.0325965x + 288.95354$ and from b), the cost function is $C(x) = 53.5031847x + 82245.2229$. The revenue function is given by $R(x) = p(x)x$ where $p(x)$ is the price demand function. The profit function is now calculated as follows:

$$
\begin{aligned}
P(x) &= R(x) - C(x) = p(x)x - C(x) \\
&= (-0.00002352x^2 - 0.0325964x + 288.95354)\,x - (53.5031847x + 82245.2229) \\
&= -0.00002352x^3 - 0.0325964x^2 + 288.95354x - 53.5031847x - 82245.2229 \\
&= -0.00002352x^3 - 0.0325964x^2 + 235.4503553x - 82245.2229
\end{aligned}
$$

To calculate maximum profit, we solve $P'(x) = 0$.

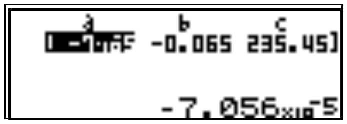
Now, $P'(x) = (3)(-0.00002352)x^2 - (2)(0.0325964)x + 235.450355$ or equivalently,

$$P'(x) = -0.00007056x^2 - 0.0651928x + 235.450355.$$

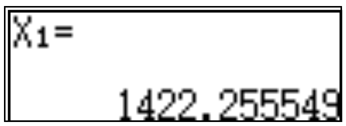
To find x such that $P'(x) = 0$ we use the following calculator commands:

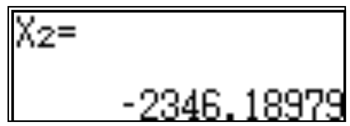
[Operations]

- Press Mode. Select 5: EQN [MODE] [5]
- Select 3, AX^2+BX+C [3]
- Enter A, $(3)(-0.00002352)$ [3] [×] [(−)] [0] [.] [0] [0] [0] [0] [2] [3] [5] [2] [=]
- Enter B, $-(2)(0.0325964)$ [(−)] [2] [×] [0] [.] [0] [3] [2] [5] [9] [6] [4] [=]
- Enter C, 235.450355 [2] [3] [5] [.] [4] [5] [0] [3] [5] [5] [=]



- Enter equals sign to get first root [=]
- Enter equals sign again to get next root [=]





The solutions to $P'(x) = -0.00007056x^2 - 0.0651928x + 235.450355 = 0$ are $x_1 = 1422.26$ and $x_2 = -2346.19$.

We do not consider a solution that is negative so we now verify $P(1422.26)$ to determine the maximum profit. We illustrate how to obtain $P(1422.26)$ using the calculator.

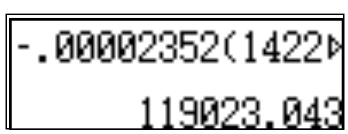
[Operations]

- Press Mode. Select 1: COMP [MODE] [1]
- Enter $-0.00002352(1422.26)^3$

  [(−)] [.] [0] [0] [0] [0] [2] [3] [5] [2] [(] [1] [4] [2] [2] [.] [2] [6] [)] [$x^\blacksquare$] [3] [▶]
- Enter $-0.0325964(1422.26)^2$

  [−] [.] [0] [3] [2] [5] [9] [6] [4] [(] [1] [4] [2] [2] [.] [2] [6] [)] [x^2]
- Enter + $235.4503553(1422.26)$

  [+] [2] [3] [5] [.] [4] [5] [0] [3] [5] [5] [3] [(] [1] [4] [2] [2] [.] [2] [6] [)]
- Enter -82245.2229 [−] [8] [2] [2] [4] [5] [.] [2] [2] [2] [9] [=]

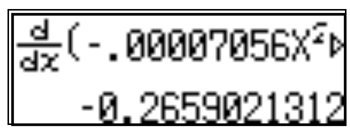


We obtain $P(1422.26) = 119{,}023.043$. Thus, the maximum profit is given to be approximately \$119,023 and this occurs when the annual demand of saws is 1422.26 units.

A way to check that maximum occurs at 1422.26 is to check the sign of the second derivative of the profit function at 1422.26, that is , to determine $P''(1422.26)$. We get $P''(1422.26) < 0$ which tells us that graph of *P* is concave down at 1422.26, and we have a maximum profit of 119,023.043.

[Operations]

- Press Mode. Select 1: COMP — MODE 1
- Select derivative — SHIFT ∫ ($\frac{d}{dx}$ ▪)
- Enter $-0.00007056x^2$ — (−) . 0 0 0 0 7 0 5 6 ALPHA) (X) x^2
- Enter $-0.0651928x$ — − . 0 6 5 1 9 5 2 8 ALPHA) (X)
- Enter $+235.450355$ — + 2 3 5 . 4 5 0 3 5 5
- Enter 1422.26 — ▶ 1 4 2 2 . 2 6 =



Activity 2 . The financial department of a company that produces automatic cameras arrived at the following revenue and cost functions:

$$R(x) = 94.8x - 5x^2 \text{ and } C(x) = 156 + 19.7x \text{ , } 1 \le x \le 15$$

a. Find the output that will produce maximum revenue.

b. Find the break even points.

c. For what outputs will a loss occur? A profit?

d. Find the output that will give a maximum profit.

Solution:

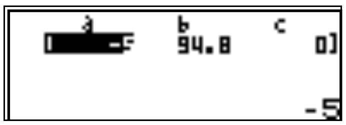
a. Using calculus we obtain maximum revenue by solving $R'(x) = 94.8 - 10x = 0$. We obtain $x = \dfrac{94.8}{10} = 9.48$. Thus, producing about 10 cameras will give maximum revenue.

Using an algebraic approach, maximum revenue will occur at an x value midway between the roots of $R(x) = 94.8x - 5x^2$.

We find the roots of $R(x) = 94.8x - 5x^2$ using the calculator as follows:

[Operations]

- Press Mode. Select 5: EQN [MODE] [5]
- Select 3, ax^2+bx+c [3]
- Enter a,-5 [(−)] [5] [=]
- Enter b, 94.8 [9] [4] [•] [8] [=]
- Enter c, 0 [0] [=]
- Enter equals sign to get first root [=]
- Enter equals sign again to get next root [=]



The roots of $R(x) = 94.8x - 5x^2$ are $x_1 = \dfrac{474.}{25} = 18.96$ and $x_2 = 0$.

Thus, maximum revenue will occur at $x = \dfrac{x_1 + x_2}{2} = \dfrac{18.96}{2} = 9.48$.

b. The break-even points are the production levels at which $R(x) = C(x)$ or equivalently, the values of x where there is no profit or $P(x) = 0$. Solving $94.8x - 5x^2 = 156 + 19.7x$ or $75.1x - 5x^2 - 156 = 0$, we find the roots of $75.1x - 5x^2 - 156$ to be: $x_1 = 2.49$ and $x_2 = 12.53$. We use the same calculator keystrokes given in a) to obtain the answers.

c. A loss occurs if $P(x) < 0$ and a profit occurs if $P(x) > 0$. Now, $P(x) > 0$ in the interval $[2.49, 12.53]$ where the break even points occur at $x_1 = 2.49$ and $x_2 = 12.53$. Moreover, $P(x) < 0$ in the intervals $[1, 2.49]$ and $[12.53, 15]$.

d. The profit function is given by $P(x) = R(x) - C(x) = 75.1x - 5x^2 - 156$. Solving $P'(x) = 75.1 - 10x = 0$, we obtain $x = 7.51$. Thus, an output of approximately 8 cameras will yield maximum profit.

Observe that the break even points are the x-intercepts of the profit function. We can verify that the maximum profit occurs at the output level midway between the break-even points. Note that $\dfrac{x_1 + x_2}{2} = \dfrac{2.49 + 12.53}{2} = 7.51$.

EXERCISES

1. The research department in a company that manufactures clock radios established the following cost and revenue functions:

$$C(x) = 160 + 10x,\ R(x) = 50x - 1.25x^2$$

a. Determine the maximum revenue and the output that produces the maximum revenue.

b. Find the break even points.
c. Does the maximum profit appear to occur at the same output level as the maximum revenue?

Solution:

a. Using calculus, $R'(x) = 50 - 2.50x = 0$ implies $x = \frac{50}{2.5} = 20$. Thus, producing about 20 radios will give maximum revenue.

b. The break-even points are obtained from $R(x) = C(x)$. Now, $50x - 1.25x^2 = 160 + 10x$ implies $40x - 1.25x^2 - 160 = 0$. The solutions to this equation are $x_1 = 27.31$ and $x_2 = 4.69$.

c. Note that $\frac{x_1 + x_2}{2} = \frac{27.31 + 4.69}{2} = 16$. Thus, maximum profit will occur at production level of 16 radios. Maximum profit does not occur at the same output where maximum revenue occurs.

Applications of Integration

We now explore some of the applications of the definite integral by using it to compute areas between curves as applications of quantities in science and business and economics.

Activity 1. The data in the table describe the distribution of wealth in the country:

x	0	0.20	0.40	0.60	0.80	1
y	0	0.12	0.31	0.54	0.78	1

a. Use a quadratic regression to find the equation of a Lorenz curve for the data
b. Approximate the index of income concentration

Solution:

a. The Lorenz curve represents the distribution of wealth in a given country. We determine this curve for the given data above using quadratic regression.

[Operations]

- Press Mode. Select 3: STAT `MODE` `3`
- Select 3: Quadratic Regression _ + CX^2 `3`
- Enter each of the data in the table- *x* values in first column, *y* values in the second column. After all of the data is entered, press `AC`.
- Calculate the regression curve A +BX + CX^2

  Calculate the constant term A `SHIFT` `1` `5` `1` `=`

  Calculate the coefficient of X, B `SHIFT` `1` `5` `2` `=`

  Calculate the coefficient of X^2, C `SHIFT` `1` `5` `3` `=`

We obtain $A = -0.15$, $B = 0.7175$ and $C = 0.3125$. The quadratic function representing the distribution of wealth is given by $f(x) = 0.3125x^2 + 0.7175x - 0.015$.

b. The ratio of the area bounded by $y = x$ and the Lorenz curve $y = f(x)$ to the area of the triangle under the line $y = x$ from $x = 0$ to $x = 1$ is called the index of income concentration. The index of income concentration is used to compare income distributions at various points in time, between groups of people, between countries, and so on.

The **index of income concentration** is given by

$$2\int_0^1 [x - f(x)]dx$$

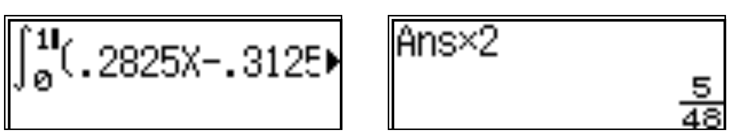
We now solve $2\int_0^1 [x - (0.3125x^2 + 0.7175x - 0.015)]dx = 2\int_0^1 [0.2825x - 0.3125x^2 + 0.015)]dx$ using the calculator. We have the following calculator keystrokes:

[Operations]

- Press Mode. Select 1: COMP — [MODE] [1]
- Select integral sign — [∫▭]

Enter the function, then the upper limit and lower limit of the integral. Navigate using the cursor keys [▼] [▲]

- Enter $(.2825x$ — [(] [•] [2] [8] [2] [5] [ALPHA] [)] (X)
- Enter $-0.3125x^2$ — [−] [•] [3] [1] [2] [5] [ALPHA] [)] (X) [x^2]
- Enter $+.015)$ — [+] [•] [0] [1] [5] [)] [▼]
- Enter lower limit, 0 — [0] [▲]
- Enter upper limit, 1 — [1] [=]
- Multiply answer by 2 — [×] [2]



The index of income concentration is approximately $\frac{5}{48} = 0.104$. The closer the index is to 0, the closer the income is to being equally distributed.

Activity 2. In a certain city the temperature (in ^0F) t hours after 9 A.M. was modeled by the function

$$T(t) = 50 + 14\sin\frac{\pi}{12}t$$

Find the average temperature during the period from 9 A.M. to 9 P.M.

Solution:

The mean value theorem for integrals states that if f is continuous on $[a,b]$ then there is a number c in $[a,b]$ such that

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

Using this result, we calculate the average temperature as

$$T_{ave} = \frac{1}{12}\int_0^{12}(50 + 14\sin\frac{\pi}{12}t)dt$$

[Operations]

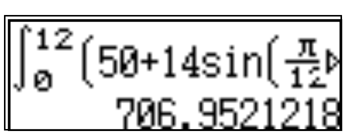
- Press SHIFT SET UP — SHIFT MODE (SETUP)

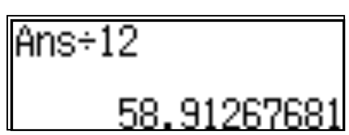
- Select 4: Rad
- Press Mode. Select 1: COMP — MODE 1
- Select integral sign — ∫

Enter the function, then the upper limit and lower limit of the integral. Navigate using the cursor keys ▼ ▲

- Enter $(50 +$ — (5 0 +
- Enter $14\sin\frac{\pi}{12}x)$ — 1 4 sin ▭ SHIFT ×10ˣ (π) ▼ 1 2 ▶ ALPHA) (X))) ▼
- Enter lower limit, 0 — 0 ▲
- Enter upper limit, 12 — 1 2 =
- Divide answer by 12 — ÷ 1 2 =

The answer we obtain is approximately 59 ^{0}F.





EXERCISES

1. A study conducted by a certain state determines that the Lorenz curves for high school teachers and real estate brokers are given respectively by the functions

$$L_1 = 0.67x^4 + 0.33x^3$$
$$L_2 = 0.72x^2 + 0.28x$$

For which profession is the distribution of income more fairly distributed?

Solution:

We use the formula in activity 1 for the index of income concentration, given by $2\int_0^1 [x - f(x)]dx$.

We follow the calculator keystrokes also given in activity 1 in solving the definite integrals.

For the high school teachers, the index of income concentration is

$$2\int_0^1 [x-(0.67x^4 + 0.33x^3)]dx = 2\int_0^1 [x-0.67x^4 - 0.33x^3]dx.$$

Using the calculator, we obtain the index 0.567.

Now, for the real estate brokers, the index of income concentration is

$$2\int_0^1 [x-(0.72x^2 + 0.28x)]dx = 2\int_0^1 [0.72x-0.72x^2]dx.$$

The answer we obtain is 0.24.

Since the index of income concentration of the real estate brokers is closer to 0, the closer the income is to being equally distributed.

2. In a certain underdeveloped country, the life expectancy of a person t years old is $L(t)$ years where

$$L(t) = \frac{110e^{0.015t}}{1+e^{0.015t}}$$

a. What is the life expectancy of a person in this country at birth? At age 50?
b. What is the average life expectancy of all people in this country between the ages of 10 and 70?
c. Find the age T such that $L(T) = T$. Call T the life limit for this country. What can be said about the life expectancy of a person older than T years?

Solution:

a. The life expectancy of a person in this country at birth is given by the value of $L(t)$ at $t = 0$. We have

$$L(0) = \frac{110e^{0.015(0)}}{1+e^{0.015(0)}} = \frac{110}{2} = 55 \text{ years old.}$$

On the other hand, the life expectancy of a person in this country at age 50 is given by $L(50)$. We solve $L(50)$ using the calculator.

[Operations]

- Press Mode. Select 1: COMP — [MODE] [1]
- Select quotient sign — [▭/▭]
- Enter $110e^{0.015(50)}$ — [1] [1] [0] [ALPHA] [×10ˣ] (e) [x▪]
- Continue — [•] [0] [1] [5] [(] [5] [0] [)] [▶] [▼]
- Enter $1+e^{0.015(50)}$ — [1] [+] [ALPHA] [×10ˣ] (e) [x▪] [•] [0] [1] [5] [(] [5] [0] [)] [=]

We have $L(50) = \dfrac{110e^{0.015(50)}}{1+e^{0.015(50)}} \approx 74.71 \approx 75$ years old.

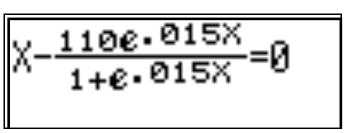
b.The average life expectancy of all people in this country between the ages of 10 and 70 is given by

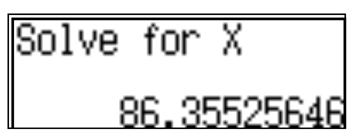
$$L_{ave} = \frac{1}{70-10}\int_{10}^{70} \frac{110e^{0.015t}}{1+e^{0.015t}}\,dt \approx 70.78 \approx 71 \text{ years old.}$$

c. We now solve the equation $t - \dfrac{110e^{0.015t}}{1+e^{0.015t}} = 0$. We use the following calculator keystrokes:

[Operations]

- Press Mode. Select 1: COMP — [MODE] [1]
- Enter x - — [ALPHA] [)] (X) [−]
- Select quotient sign — [▭/▭]
- Enter $110e^{0.015x}$ — [1] [1] [0] [ALPHA] [×10ˣ] (e) [x▪] [•] [0] [1] [5] [ALPHA] [)] (X) [▶] [▼]
- Enter $1+e^{0.015x}$ — [1] [+] [ALPHA] [×10ˣ] (e) [x▪] [•] [0] [1] [5] [ALPHA] [)] (X) [▶] [▶]
- Enter = 0 — [ALPHA] [CALC] (=) [0]
- Select solve equation — [SHIFT] [CALC] (SOLVE)





The answer we obtain is approximately 86.36 years. They have already exceeded their life expectancy.

4. Geometry

Transformation Geometry

Matrices are used in transformation geometry.

For example, if a point $A(x, y)$ is rotated counterclockwise through an angle θ about the origin, its new coordinates $B(x', y')$ can be obtained by applying the transformation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ as follows:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

The following examples illustrate how to obtain the transformed image of a given triangle under rotations and translations.

Activity 1. a. Find the image of the triangle ΔABC under the 45° counterclockwise rotation about the origin, with $A(2,0), B(4,4)$ and $C(5,2)$.

b. Find the image of the triangle ΔABC under the 90°, 135° counterclockwise rotations.

c. Find the image of the triangle ΔABC under the 45° clockwise rotation.

Solution:

a.We apply the rotation matrix T = $\begin{bmatrix} \cos 45^{\circ} & -\sin 45^{\circ} \\ \sin 45^{\circ} & \cos 45^{0} \end{bmatrix}$ to the points $A(2,0)$, $B(4,4)$ and $C(5,2)$ then determine the vertices $A'(x_1', y_1')$, $B'(x_2', y_2')$ and $C'(x_3', y_3')$ of the transformed triangle under the rotation. We enter the x-coordinates and y-coordinates of A, B, C in the first and second rows respectively of a 2 x 3 matrix. We have

$$\begin{bmatrix} \cos 45^{\circ} & -\sin 45^{\circ} \\ \sin 45^{\circ} & \cos 45^{0} \end{bmatrix} \begin{bmatrix} 2 & 4 & 5 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} x_1' & x_2' & x_3' \\ y_1' & y_2' & y_3' \end{bmatrix}$$

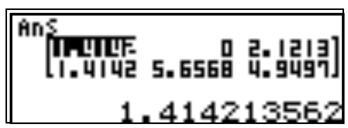
We use the calculator to solve the coordinates of the vertices A', B', C'.

[Operations]

- Select Mode 1:COMP — [MODE] [1]
- Select Degree Mode — [SHIFT] [MODE] (SETUP) [3]
- Press Mode. Select 6: Matrix — [MODE] [6]
- Access Matrix A — [1]
- Select 2x2 matrix — [5]

• Enter cos 45^0,-sin 45^0	COS 4 5 = (−) sin 4 5 =
• Enter sin 45^0,cos 45^0	sin 4 5 = COS 4 5 =
• Access Matrix B	SHIFT 4 1 2
• Select 2x3 matrix	4
• Enter first row 2,4,5	2 = 4 = 5 =
• Enter second row 0,4,2	0 = 4 = 2 =
• Press AC	AC
• Enter MatA	SHIFT 4 3
• Enter x MatB	× SHIFT 4 4 =

The answer we obtain is the matrix $\begin{bmatrix} 1.4142 & 0 & 2.1213 \\ 1.4142 & 5.6568 & 4.9497 \end{bmatrix}$.



Thus the vertices of the transformed triangle are $A'(1.4,1.4), B'(0,5.7), C'(2.1,4.9)$. (Note that the values have been rounded off to four decimal places.)

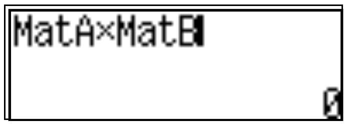
b.To get the transformed triangle under a 90^0 counterclockwise rotation, we follow the same process given in (i). We apply the transformation matrix $\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^0 \end{bmatrix}$ to $A(2,0), B(4,4)$ and $C(5,2)$.

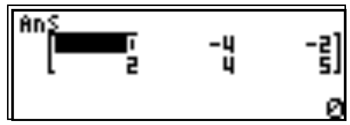
[Operations]

• Press Mode. Select 6: Matrix	MODE 6
• Access Matrix A	1
• Select 2x2 matrix	5
• Enter cos 90^0,-sin 90^0	COS 9 0 = (−) sin 9 0 =
• Enter sin 90^0,cos 90^0	sin 9 0 = COS 9 0 =
• Access Matrix B	SHIFT 4 1 2
• Select 2x3 matrix	4
• Enter first row 2,4,5	2 = 4 = 5 =
• Enter second row 0,4,2	0 = 4 = 2 =
• Press AC	AC
• Enter MatA	SHIFT 4 3
• Enter x MatB	× SHIFT 4 4 =

Using the calculator, we obtain the following result:

$$\begin{bmatrix}\cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^0\end{bmatrix}\begin{bmatrix}2 & 4 & 5 \\ 0 & 4 & 2\end{bmatrix}=\begin{bmatrix}0 & -4 & -2 \\ 2 & 4 & 5\end{bmatrix}$$





The learner can explore the notion that we can also obtain the image of ΔABC under a 90^0 rotation by applying two consecutive 45^0 rotations to ΔABC or applying T^2 to ΔABC. This is the same as applying T to the transformed triangle given in (i) which is $\Delta A'B'C'$ having vertices $A'(1.4,1.4), B'(0,5.7), C'(2.1,4.9)$.

We have the following result using the calculator:

$$\begin{bmatrix}\cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^0\end{bmatrix}\begin{bmatrix}\cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^0\end{bmatrix}\begin{bmatrix}2 & 4 & 5 \\ 0 & 4 & 2\end{bmatrix}$$

$$=\begin{bmatrix}\cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^0\end{bmatrix}\begin{bmatrix}1.4142 & 0 & 2.1213 \\ 1.4142 & 5.6568 & 4.9497\end{bmatrix}=\begin{bmatrix}0 & -3.999 & -1.999 \\ 1.999 & 3.999 & 4.999\end{bmatrix}.$$

Note that we obtain the same result as the matrix given earlier.

Now, to get the image of ΔABC under the 135° counterclockwise rotation, we apply T^3 or apply matrix

$$\begin{bmatrix}\cos 135^\circ & -\sin 135^\circ \\ \sin 135^\circ & \cos 135^0\end{bmatrix} \text{ to } \begin{bmatrix}2 & 4 & 5 \\ 0 & 4 & 2\end{bmatrix}.$$

We obtain $\begin{bmatrix}-1.4142 & -5.656 & -4.949 \\ 1.4142 & 0 & 2.1213\end{bmatrix}$ using the calculator.

c. Suppose triangle $\Delta A'B'C'$ with $A'(x_1', y_1'), B'(x_2', y_2')$ and $C'(x_3', y_3')$ is the image of triangle ΔABC with $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ under the 45° clockwise rotation, α, that is, we have

$$\begin{bmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3\end{bmatrix}=\begin{bmatrix}x_1' & x_2' & x_3' \\ y_1' & y_2' & y_3'\end{bmatrix}$$

Observe that applying a 45° counterclockwise rotation to $\Delta A'B'C'$ will give us ΔABC.

Thus,

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^0 \end{bmatrix} \begin{bmatrix} x_1' & x_2' & x_3' \\ y_1' & y_2' & y_3' \end{bmatrix}$$

This implies

$$\begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^0 \end{bmatrix}^{-1} \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} x_1' & x_2' & x_3' \\ y_1' & y_2' & y_3' \end{bmatrix} \text{ or}$$

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^0 \end{bmatrix}^{-1}$$

Thus to apply a 45° clockwise rotation, α, is the same as applying the rotation matrix T^{-1}.

We now calculate $T^{-1} = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^0 \end{bmatrix}^{-1}$ using the calculator.

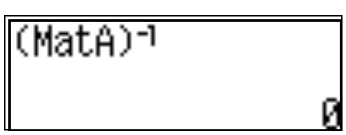
.

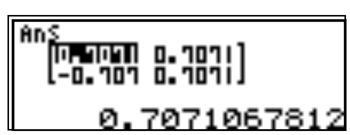
[Operations]

- Press Mode. Select 6: Matrix — [MODE] [6]
- Access Matrix A — [1]
- Select 2x2 matrix — [5]
- Enter cos 45^0, -sin 45^0 in first row — [cos] [4] [5] [=] [(−)] [sin] [4] [5] [=]
- Enter sin 45^0, cos 45^0 in second row — [sin] [4] [5] [=] [cos] [4] [5]
- Press AC — [AC]
- Enter parenthesis — [(]
- Access the matrix — [SHIFT] [4] [3]
- Access parenthesis — [)]
- Access inverse — [x^{-1}] [=]

We obtain the following matrix:

$$\begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^0 \end{bmatrix}^{-1} = \begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}$$





So we have $\begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$

$$= \begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}\begin{bmatrix} 2 & 4 & 5 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1.4142 & 5.6568 & 4.9497 \\ -1.4142 & 0 & -2.121 \end{bmatrix}.$$

Thus the vertices of the transformed triangle are $A'(1.4,-1.4), B'(5.7,0), C'(4.9,-2.1)$.

Activity 2. Translate ΔABC with vertices $A(2.5,0.75)$, $B(5.2,-1.5)$ and $C(6.1,4.3)$ five units to the right and three units down. Find the coordinates of the transformed triangle.

Solution:

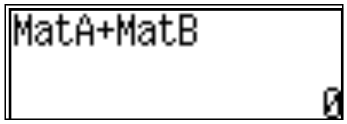
Let $A'(x_1, y_1)$, $B'(x_2, y_2)$ and $C'(x_3, y_3)$ be the coordinates of the transformed triangle under translation. We enter the x-coordinates and y-coordinates of A, B, C in the first and second rows respectively of a 2 x 3 matrix. Then we add 5 to the x coordinates and add -3 to the y-coordinates. We have the following equation involving matrices:

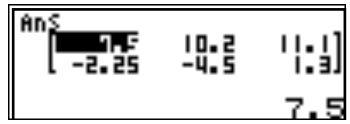
$$\begin{bmatrix} 2.5 & 5.2 & 6.1 \\ 0.75 & -1.5 & 4.3 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ -3 & -3 & -3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

[Operations]

- Press Mode. Select 6: Matrix [MODE] [6]
- Access Matrix A [1]
- Select 2x3 matrix [4]
- Enter first row 2.5,5.2,6.1 [2] [•] [5] [=] [5] [•] [2] [=] [6] [•] [1] [=]
- Enter second row 0.75,-1.5,4.3 [•] [7] [5] [=] [(−)] [1] [•] [5] [=] [4] [•] [3] [=]
- Access Matrix B [SHIFT] [4] [1] [2]
- Select 2x3 matrix [4]
- Enter first row 5 [5] [=] [5] [=] [5] [=]
- Enter second row -3 [(−)] [3] [=] [(−)] [3] [=] [(−)] [3] [=]
- Press AC [AC]
- Enter MatA [SHIFT] [4] [3]
- Enter + MatB [+] [SHIFT] [4] [4] [=]

The answer we obtain is the matrix $\begin{bmatrix} 7.5 & 10.2 & 11.1 \\ -2.25 & -4.5 & 1.3 \end{bmatrix}$.





The coordinates of the transformed triangle are $A'(7.5,-2.25), B'(10.2.-4.5), C'(11.1,1.3)$.

EXERCISES

1. One vertex of a regular pentagon centered at the origin is $A(2,0)$. Find the coordinates of the other four vertices.

2. What transformation matrix will be used?

Solution:

We apply the rotation matrix T = $\begin{bmatrix} \cos 72^\circ & -\sin 72^\circ \\ \sin 72^\circ & \cos 72^0 \end{bmatrix}$ to carry out a 5-fold rotation.

Our four other vertices are found to be

T: $\begin{bmatrix} \cos 72^\circ & -\sin 72^\circ \\ \sin 72^\circ & \cos 72^0 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.236. \\ 3.8042 \end{bmatrix}$

T^2: $\begin{bmatrix} \cos 72^\circ & -\sin 72^\circ \\ \sin 72^\circ & \cos 72^0 \end{bmatrix} \begin{bmatrix} \cos 72^\circ & -\sin 72^\circ \\ \sin 72^\circ & \cos 72^0 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -3.236. \\ 2.3511 \end{bmatrix}$

T^3 : $\begin{bmatrix} -3.236. \\ -2.3511 \end{bmatrix}$

T^4: $\begin{bmatrix} 1.236 \\ -3.804 \end{bmatrix}$

5. Complex Numbers

Geometric Representation of Complex Numbers

Each number on the complex plane can be represented in terms of its distance from the origin or pole and the angle the line segment makes with the positive real axis. The length of the line is called the **amplitude** or the **modulus**. The angle is called the **argument** of the complex number.

In the first part of the chapter, we show how the calculator can be a helpful tool in understanding the geometric representation of complex numbers. The first activity shall investigate some properties pertaining to the argument of complex numbers.

ACTIVITY

For a complex number $z = a + bi$, with argument θ, determine the argument of the following complex numbers:

a. $2z$, $3z$,…, nz (n a positive integer)
b. $-z$
c. z^2, z^3

Consider the following representative set of complex numbers in the exploration: $1+i$, $-1+\sqrt{3}i$ and $\sqrt{2}-\sqrt{2}i$.(Note: Other examples of complex numbers may be used)

Solution:

The arguments of the complex numbers can be obtained using the calculator in the manner given below.

We illustrate for the complex number $1+i$.

[Operations]

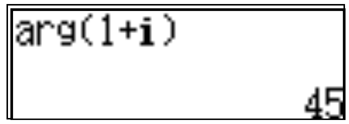
A. Approach 1

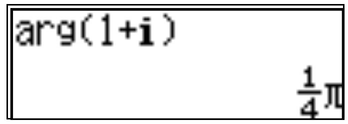
- Press Mode. Select 2: CMPLX — MODE 2
- Press Math format. — SHIFT MODE (SETUP) 1

(Throughout this note, we will use the complex mode and math format.)

Enter degree or radian mode:

- To obtain argument in degree — SHIFT MODE 3
- To obtain argument in radian — SHIFT MODE 4
- Access argument symbol — SHIFT 2 1
- Enter $1+i$ — 1 + SHIFT ENG) =

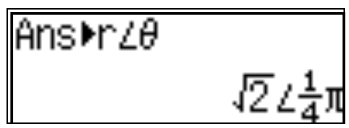




B. Approach 2

Note: you may also obtain the argument of the complex number, together with the modulus by accessing the polar form of the number as follows:

- Enter $1+i$ — [1] [+] [SHIFT] [ENG] [)] [=]
- Access polar form — [SHIFT] [2] [3] [=]



C. Approach 3

Or a shorter way to determine the argument and modulus is to enter a command so that the calculator can return the polar form outright whenever you enter a complex number.

- Press SHIFT SET UP — [SHIFT] [MODE] (SETUP) [▼]
- Enter 3 CMPLX — [3] [2]
- Enter $1+i$ — [1] [+] [SHIFT] [ENG] [)] [=]

Observe that the argument of $1+i$ is $\frac{\pi}{4}(45^o)$.

We repeat the process for $-1+\sqrt{3}i$ and $\sqrt{2}-\sqrt{2}i$.

We find that the argument of $-1+\sqrt{3}i$ is $\frac{2\pi}{3}(120^o)$ and the argument of $\sqrt{2}-\sqrt{2}i$ is

$-\frac{\pi}{4}(-45^o)$.

a. We now verify the arguments of $2z$, $3z$,…, nz. We have the following results:

z	argument z	argument $2z$	argument $3z$
$1+i$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$
$-1+\sqrt{3}i$	$\frac{2\pi}{3}$	$\frac{2\pi}{3}$	$\frac{2\pi}{3}$
$\sqrt{2}-\sqrt{2}i$	$-\frac{\pi}{4}$	$-\frac{\pi}{4}$	$-\frac{\pi}{4}$

Observe that nz and z have the same argument.

b. Next, we check the arguments of $-z$. We obtain the following results:

z	argument z	argument $-z$
$1+i$	$\frac{\pi}{4}$	$-\frac{3\pi}{4}$
$-1+\sqrt{3}i$	$\frac{2\pi}{3}$	$-\frac{\pi}{3}$
$\sqrt{2}-\sqrt{2}i$	$-\frac{\pi}{4}$	$3\frac{\pi}{4}$

Note that the argument of z and argument of $-z$ differ by π. In fact, argument $-z = \theta + \pi$.

c. Now, we determine the arguments of z^2, z^3.

z	argument z	argument z^2	argument z^3
$1+i$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$
$-1+\sqrt{3}i$	$\frac{2\pi}{3}$	$-\frac{2\pi}{3}$	0
$\sqrt{2}-\sqrt{2}i$	$-\frac{\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{3\pi}{4}$

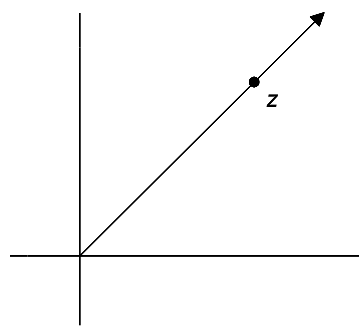
The argument of z^2 is 2θ. Similarly, the argument of z^3 is 3θ.

EXERCISES

1. On a complex plane, describe the graph of **argument** $z = \frac{\pi}{4}$. Describe the set of complex numbers that satisfies this equation.

Solution:

From the exploratory activity, nz and z have the same argument for n a positive integer. In fact this is also true for n any positive real. The set of complex numbers satisfying **argument** $z = \frac{\pi}{4}$ is $\{n(1+i)$: n any positive real$\}$. The graph represents a half-line making an angle of $\frac{\pi}{4}$ with the positive x-axis.

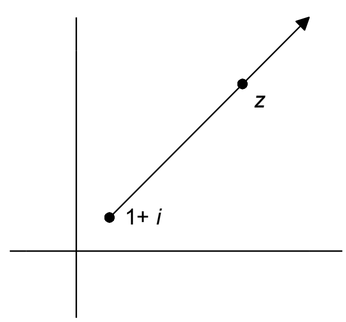


2. On a complex plane, sketch the loci of **argument** $(z-1-i) = \frac{\pi}{4}$.

Solution:

We can write argument $(z-1-i) = \frac{\pi}{4}$ as argument $(z-(1+i) = \frac{\pi}{4}$.

Letting $z = z_1 + (1+i)$, we have argument $(z_1) = \frac{\pi}{4}$. From Exercise 1, z_1 belongs to the set, $\{n(1+i)$: n any positive real$\}$. In which case, z will be any element in $\{n(1+i)+(1+i)$: n any positive real$\}$. The graph is given as:



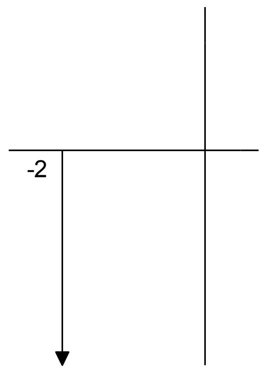
Using the calculator, we can take representative elements of this set to verify our answer, example: 2+2i, 3+3i, etc.

3. On a complex plane, sketch the loci of **argument** $(z+2) = \frac{3\pi}{2}$.

Solution:

Notice that we can explore that any number in the set $\{-ni$:for n any real number$\}$ has argument $\frac{3\pi}{2}$.

The elements of the form $-ni-2$ will satisfy **argument** $(z+2) = \frac{3\pi}{2}$. We have the graph below.



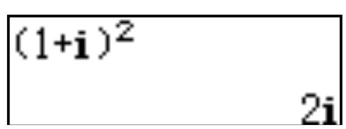
4.If $z = 1+i$, mark on an Argand Diagram the four points A, B, C and D representing z, z^2, z^3, z^4

Solution:

The following keystrokes gives us the $a+bi$ form of z, z^2, z^3, z^4.

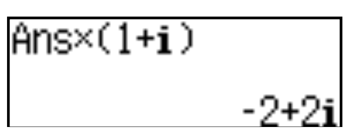
[Operations]

- Press Mode. Select 2: CMPLX — [MODE] [2]
- Press Math format — [SHIFT] [MODE] (SETUP) [1]
- Enter $(1+i)^2$ — [(] [1] [+] [SHIFT] [ENG] [)] [x^2] [=]



We obtain 2i.

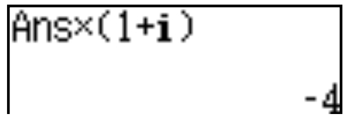
- Enter ANS x (1+i) — [Ans] [×] [(] [1] [+] [SHIFT] [ENG] [)] [=]



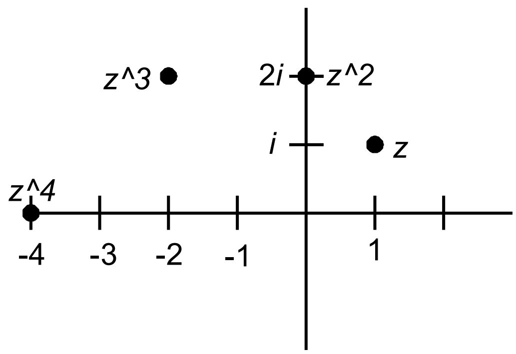
We obtain -2+ 2i.

- Enter ANS x (1+i)

We obtain -4.



We have the following points on the Argand Plane.



Note that we can verify our results from the exploratory activity: as seen in the Argand diagram, the arguments of z^2, z^3 and z^4 are respectively $\frac{\pi}{4}, \frac{\pi}{2}, 3\frac{\pi}{4}$ and π.

Magnitude of Complex Numbers

Complex numbers are useful in many practical applications, for instance, in describing electricity and electronic devices.

In this section we show how the scientific calculator can be used to explore notions on the magnitude of complex numbers and solve related real-world problems.

ACTIVITY

For a complex number $z = a + bi$, verify the following properties:

a. $|z| = |-z|$

b. $|z| = |\bar{z}|$

c. $|z| = |-\bar{z}|$

where we define the **magnitude** or **modulus** of any complex number z as $|z| = |a + bi| = \sqrt{a^2 + b^2}$.
The magnitude or modulus is generally represented with the absolute value symbol, $|\ |$. Sometimes, as on the calculator, it is called the **absolute value** of the complex number.

We use the following representative set of complex numbers in the exploration: $\sqrt{3} - i$, $-1 + \sqrt{3}i$ and $\sqrt{2} + \sqrt{2}i$.(Note: Other examples of complex numbers may be used)

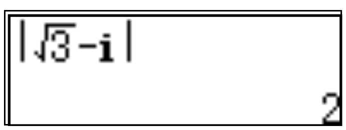
a. We verify $|z| = |-z|$ using $z = \sqrt{3} - i$.

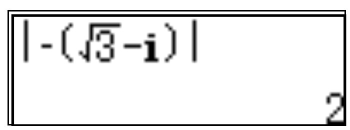
[Operations]

- Press Mode. Select 2: CMPLX [MODE] [2]
- Press Math format [SHIFT] [MODE] (SETUP) [1]

(Throughout this note, we will use the complex mode and math format.)

- Access absolute value symbol [SHIFT] [hyp] (Abs)
- Enter $\sqrt{3} - i$ [√■] [3] [▶] [−] [SHIFT] [ENG]
- Access absolute value symbol [SHIFT] [hyp] (Abs)
- Enter $-(\sqrt{3} - i)$ [(−)] [(] [√■] [3] [▶] [−] [SHIFT] [ENG] [)]





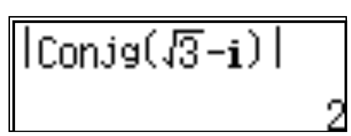
Observe that we obtain the value $|\sqrt{3} - i| = |-(\sqrt{3} - i)| = 2$.

We repeat the same process, this time using the complex numbers $-1 + \sqrt{3}i$ and $\sqrt{2} + \sqrt{2}i$.
We obtain $|-1 + \sqrt{3}i| = |-(1 + \sqrt{3}i)| = 2$. Similarly, $|\sqrt{2} + \sqrt{2}i| = |-(2 + \sqrt{2}i)| = 2$.

b. Next, we verify $|z| = |\bar{z}|$ using the same complex numbers.

[Operations]

- Access absolute value symbol [SHIFT] [hyp] (Abs)
- Extract conjugate of $\sqrt{3} - i$. Select 2 Conjg. [SHIFT] [2] [2] [√■] [3] [▶] [−] [SHIFT] [ENG] [)]



We obtain $|\sqrt{3} - i| = |\overline{\sqrt{3} - i}| = 2$.

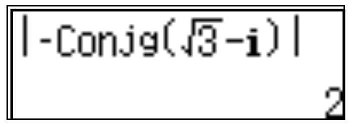
Now for the complex numbers $-1 + \sqrt{3}i$ and $\sqrt{2} + \sqrt{2}i$, it can also be shown that

$\left|-1+\sqrt{3}i\right| = \left|\overline{-1+\sqrt{3}i}\right| = 2$ and $\left|\sqrt{2}+\sqrt{2}i\right| = \left|\overline{\sqrt{2}+\sqrt{2}i}\right| = 2$.

c. Now, we look at $|z| = |-\bar{z}|$.

[Operations]

- Access absolute value symbol — SHIFT hyp (Abs)
- Extract negative conjugate of $\sqrt{3}-i$ — (−) SHIFT 2 2 √■ 3 ▶ − SHIFT ENG)



We get the result $\left|-(\overline{\sqrt{3}-i})\right| = 2$. For the other complex numbers, we obtain $\left|-(\overline{-1+\sqrt{3}i})\right| = 2$ and $\left|-(\overline{\sqrt{2}+\sqrt{2}i})\right| = 2$.

To understand more clearly the computations above, we consider their graphical representations .

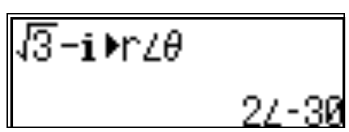
Each number on the complex plane can be represented in terms of its distance from the origin or pole and the angle the line segment makes with the positive real axis. The length of the line is called the **amplitude** or the **modulus**. The angle is called the **argument** of the complex number. Another form of a complex number is its polar form, defined in terms of its amplitude and argument.

Note for instance the complex number $z = \sqrt{3} - i$.

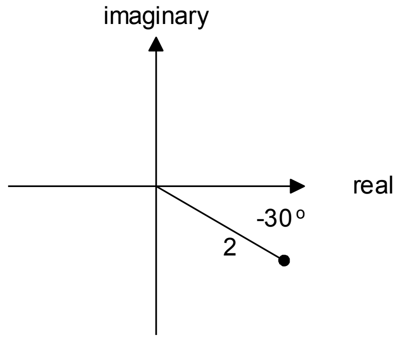
The calculator can return the amplitude as well as the argument of the complex number as follows:

[Operations]

- Enter $\sqrt{3}-i$ — √■ 3 ▶ − SHIFT ENG
- Calculate amplitude, argument — SHIFT 2 3 =

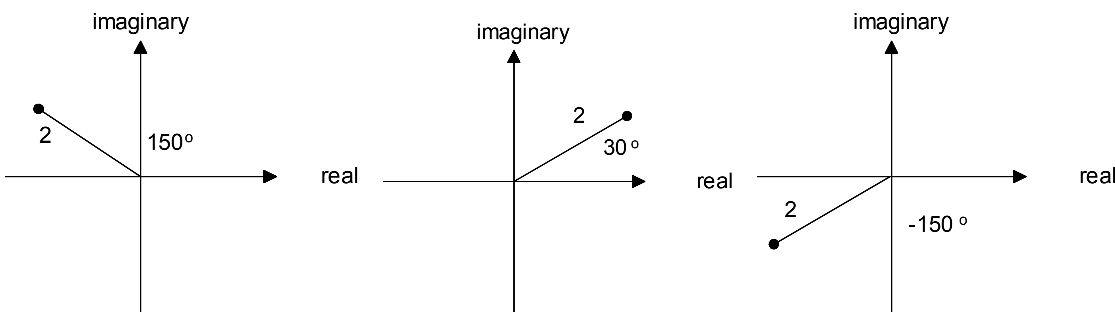


We obtain amplitude 2 and argument, -30°. The complex number $z = \sqrt{3} - i$ in polar form may be written as $[2, -30^{\circ}]$. In the complex plane, $z = \sqrt{3} - i$ is graphed as follows:



Similarly from the calculator, the polar forms of $-z = -\sqrt{3} + i$, $\bar{z} = \overline{\sqrt{3} - i} = \sqrt{3} + i$ and $-\bar{z} = -(\overline{\sqrt{3} - i}) = -(\sqrt{3} + i) = -\sqrt{3} - i$ are respectively, $[2, 150^\circ]$, $[2, 30^\circ]$ and $[2, -150^\circ]$.

In the complex plane, $-z = -\sqrt{3} + i$, $\bar{z} = \sqrt{3} + i$ and $-\bar{z} = -\sqrt{3} - i$ are represented as follows:



Graphically, the complex number z and its negative, $-z$ are reflections of each other about the origin or pole. On the other hand, z and its conjugate, $\bar{z}$ are reflections of each other across the real axis while z and its negative conjugate, are reflections of each other across the imaginary axis.

Through the graphs, we can visualize that $|z| = |-z| = |\bar{z}| = |-\bar{z}|$ for the given complex number.

To prove this result algebraically, we have

$$|z| = |a + bi| = \sqrt{a^2 + b^2} = \sqrt{(-a)^2 + (-b)^2} = |-a - bi| = |-(a + bi)| = |-z|$$

$$|z| = |a + bi| = \sqrt{a^2 + b^2} = \sqrt{(a)^2 + (-b)^2} = |a - bi| = |\overline{a + bi}| = |\bar{z}|$$

$$|z| = |a + bi| = \sqrt{a^2 + b^2} = \sqrt{(-a)^2 + (b)^2} = |-a + bi| = |-(\overline{a + bi})| = |-\bar{z}|$$

EXERCISES

1. On a complex plane, graph $|z| = 2$. Describe the set of complex numbers that satisfies this equation.

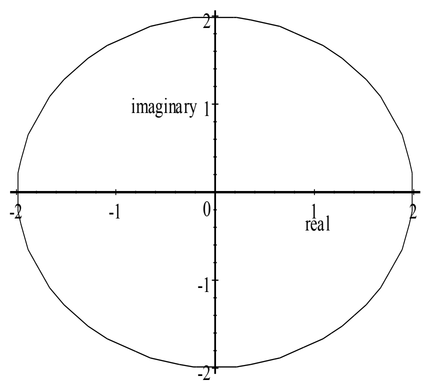
Solution:

The set of complex numbers $z : |z| = \sqrt{a^2 + b^2} = 2$ satisfying the given equation is

$$\{ z : |z| = \sqrt{a^2 + b^2} = 2 \} \text{ or equivalently } \{ z : |z| = a^2 + b^2 = 4 \}.$$

From the exploratory activity, this set will have as elements all $z = a + bi$ satisfying $a^2 + b^2 = 4$ including $-z$, $\bar{z}$ and $-\bar{z}$. The graph will be symmetric about the origin or pole, real and imaginary axes. The graph will consist of all points with distance from the origin 2. The graph is a circle centered at the origin with radius 2.

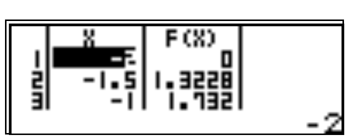
The complex numbers $\sqrt{3} - i$, $-1 + \sqrt{3}i$ and $\sqrt{2} + \sqrt{2}i$ given earlier are elements of this set. The graph is given below:

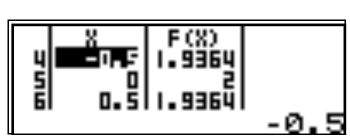


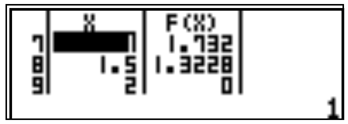
To approximate the other complex numbers $z = a + bi$ in the set, we generate the values (a, b) using the table function of the calculator. Let $a = x$, $b = f(x) = \pm\sqrt{4 - x^2}$.

[Operations]

- Press Mode. Select 7: TABLE [MODE] [7]
- Enter $f(x) = \sqrt{4 - x^2}$ [√■] [4] [−] [ALPHA] [ )] (X) [x^2]
- Start with the value -2. [(−)] [2]
- End with the value 2. [2]
- Use 'Step' .5 (one may vary this) [•] [5]







Repeat with $f(x) = -\sqrt{4-x^2}$.

2. In a particular electronic circuit, the voltage, V is given by $V = 7.375 - 4.150i$ volts and the impedance, Z is $Z = 3.145 + 2.112i$ ohms. Find the magnitude of the current, I , across this part of the circuit, using the equation, $V = IZ$.

Solution:

We will use the calculator to solve the exercise.

$$V = IZ$$

Substituting the given values for V and Z , we obtain the equation

$$7.375 - 4.150i = I(3.145 + 2.112i)$$

Solving for I , we have

$$I = \frac{7.345 - 4.150i}{3.145 + 2.112i}$$

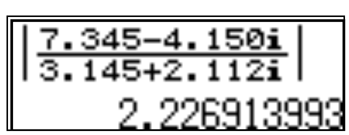
The magnitude of the current, I is given by $|I|$.

[Operations]

Enter the expression $\left|\dfrac{7.345 - 4.150i}{3.145 + 2.112i}\right|$.

- Access the absolute value symbol [SHIFT] [hyp] (Abs)
- Access the quotient symbol 
- Enter $7.345 - 4.150i$ in the numerator [7] [•] [3] [4] [5] [−] [4] [•] [1] [5] [SHIFT] [ENG]
- Enter $3.145 + 2.112i$ in the denominator

[3] [•] [1] [4] [5] [+] [2] [•] [1] [1] [2] [SHIFT] [ENG]



We obtain the magnitude of the current $|I|$, to be approximately 2.227 amperes.

7. Statistics

Mean and Standard Deviation

The mean is a measure of central tendency calculated by dividing the sum of all values by the number of values in the data set. The standard deviation is a measure of the spread that is given by the positive square root of the variance.

The interpretation of the mean and the standard deviation for a given set of data is easier when technology is present since calculations of these measures are faster and more systematic.

Consider the following problem:

ACTIVITY

A local golf club had a tournament recently. The following data gives the scores for a round of 18 holes of golf for 17 men and 15 women randomly selected from the men's and women's leagues respectively.

Men	86	67	91	78	82	66	70	91	111
	74	76	101	78	77	84	74	71	
Women	100	99	86	94	97	80	116	106	102
	96	89	99	98	93	93			

Compute the mean and standard deviation for each sample. How do they compare? What do you observe regarding the results obtained?

Solution:

The **mean** $\bar{x}$ for an ungrouped sample data is given by

$$\bar{x} = \frac{\sum x}{n}$$

where n is the sample size and $\sum x$ is the sum of all values.

The sample standard deviation s is given by

$$s = \sqrt{s^2}$$

where $s^2 = \dfrac{\sum x^2 - \dfrac{(\sum x)^2}{n}}{n-1}$ is the sample variance.

We compute the mean and standard deviation for each of the two sets of golf scores using the calculator.

[Operations]

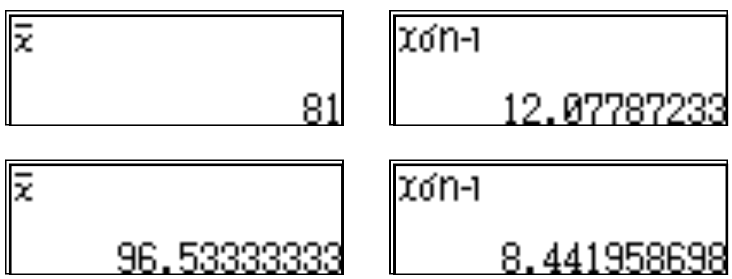
- Press Mode. Select 3: STAT [MODE] [3]
- Select 1: 1-VAR [1]
- Enter each of the data in the table then press [AC]

- Calculate the mean $\bar{x}$ [SHIFT] [1] [4] [2]
- Calculate the standard deviation $x\sigma\ n$-1 [SHIFT] [1] [4] [4]

Whenever we want to refer to the same data again [SHIFT] [1] [2]
The data will be kept in the calculator's memory.

We obtain the following calculations:

For the scores of the men's league we have mean $\bar{x} = 81$ and standard deviation $s = 12.078$. On the other hand for the women's league, we have mean $\bar{x} = 96.533$ and standard deviation $s = 8.441$.



The mean results show that the average score for men in golf is lower, $\bar{x} = 81$ than that of the women, $\bar{x} = 96.533$. These results indicate that for the sample scores obtained during the tournament, the men's average score was better than of the women's.

The standard deviation, $s = 12.078$ obtained from the men's scores is higher than that obtained from the women's scores, $s = 8.441$. This suggests a more heterogeneous sampling of the men's scores – the scores are spread over a relatively larger range around the mean.

On the other hand, there is a more homogenous sampling of the women's scores – the scores are spread over a relatively smaller range around the mean.

ACTIVITIES

1. The following data give the weights (in pounds) of a random sample of 36 college students, males (M) and females (F):

Males	173	192	180	135	192	167	178	189	139
	190	200	130	175	160	181	159	168	124
Females	100	136	99	147	102	157	163	106	113
	110	90	123	121	119	145	95	103	120

Compute the mean and standard deviation for the weights of

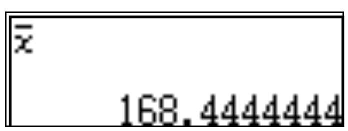
a. men only
b. women only
c. all students. Compare the measures obtained.

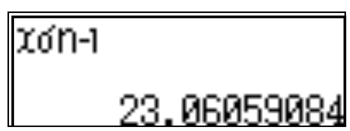
Solution:

We follow the steps given in the previous activity to calculate the mean and standard deviation using the calculator.

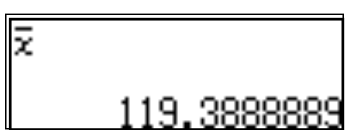
We first enter all male weights in table to answer item a. Then we enter all the female weights to answer item b. Lastly, we enter all weights (both males and females) to answer question c.

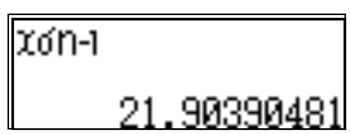
a. For male students we have $\overline{x_1} = 168.44$, $s_1 = 23.06$.



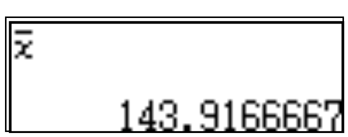


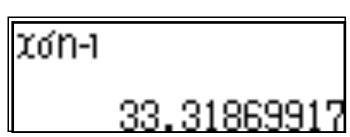
b. For female students we obtain $\overline{x_2} = 119.38$, $s_2 = 21.90$.





c. For all students we obtain $\overline{x} = 143.91$, $s = 33.32$.





The average weight for the male students $\overline{x_1}$ is heavier than the average weight of the female students, $\overline{x_2}$. For the whole population, the average weight $\bar{x} = 143.91$ is the average of $\overline{x_1} = 168.44$ and $\overline{x_2} = 119.38$; a value midway between $\overline{x_1}$ and $\overline{x_2}$. $\bar{x}$ is also called the combined mean of both data sets.

One property of the mean is that if we know the means ($\overline{x_1}$, $\overline{x_2}$) and sample sizes (n_1, n_2) of two or more data sets, we can calculate the combined mean of both data sets. The combined mean can be calculated using the formula

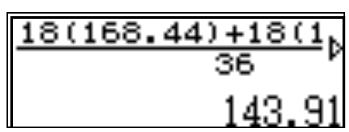
$$\bar{x} = \frac{n_1 \overline{x_1} + n_2 \overline{x_2}}{n_1 + n_2}$$

We use the following calculator keystrokes, applying the formula:

[Operations]

- Press Mode. Select 1: COMP — MODE 1
- Press Shift Setup — SHIFT MODE (SETUP)

- Select MthIO — 1
- Enter quotient sign — ▭/▭
- Enter 18(168.44) — 1 8 (1 6 8 · 4 4)
- Enter +18(119.38) — + 1 8 (1 1 9 · 3 8) ▶
- Enter 36 — 3 6 =
- Get decimal equivalence — S⇔D



We will obtain $\bar{x} = \dfrac{18(168.44) + 18(119.38)}{36} = 143.91$, which is the same answer we calculated in c.

The standard deviation obtained from the male weights is almost the same as that obtained from the female weights $s_1 = s_2 \approx 22$. The weights are clustered slightly around the mean. The standard deviation for the combined weights s is higher, suggesting that the weights are more spread around the mean.

2. Consider the following data:

66. 9	66. 2	71	68. 6	65. 4	68. 4	71. 9	**DATA D**

a. Compute the mean and standard deviation for the data.
b. Add a fixed number say 5 and 10 to all measurements in the given data set as follows:

71.9	71.2	76	73.6	70.4	73.4	76.9	**(DATA D) +5**

76.9	76.2	81	78.6	75.4	78.4	81.9	**(DATA D)+10**

Calculate the mean and standard deviation for each given data.

c. Referring to item b above, what do you observe regarding the values of the mean and standard deviation obtained? How do these compare with the mean and standard deviation obtained in item a? What generalization can be made?

d. Multiply a fixed number say 5 and 10 to all measurements in the given data set as follows:

334.5	331	355	343	327	342	359.5	**5*(DATA D)**

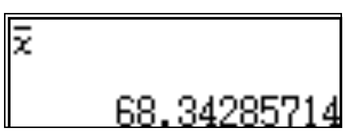
669	662	710	686	654	684	719	**10*(DATA D)**

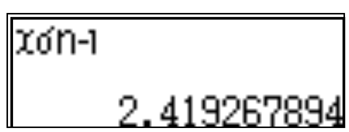
Calculate the mean and standard deviation for each given data.

e. What observation can you make regarding the new mean and standard deviation obtained from d as compared to that obtained in a?

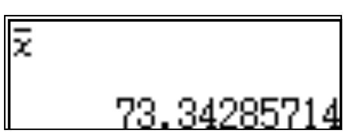
Solution:

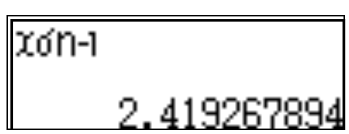
a. We obtain the following calculations for Data D above: mean $\bar{x} = 68.342$ and standard deviation $s = 2.4192$.



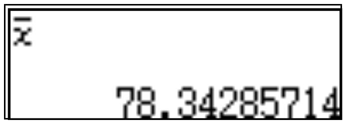


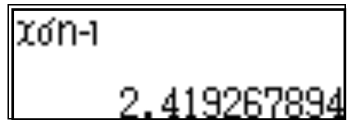
b. (DATA D) + 5





(DATA D) + 10





The mean and standard deviation obtained are as follows:

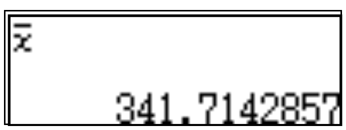
(DATA D) +5 : mean $\bar{x} = 73.342$ and standard deviation $s = 2.4192$.
(DATA D) +10 : mean $\bar{x} = 78.342$ and standard deviation $s = 2.4192$.

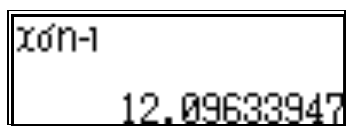
c. If 5 is added to all measurements in the given data, the value of the mean also increases by 5. Similarly, if 10 is added to all entries in the given data, the value of the mean also increases by 10.

The generalization that can be made is that if a fixed number k is added to all measurements in the data set, then the mean of the new measurements is $k + \bar{x}$, the original mean. However, the standard deviations of the new measurements remain unchanged. Note that the deviations $x - \bar{x}$ remain unchanged, thus, consequently, s^2 and s also remain unchanged.

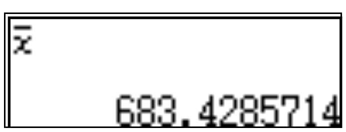
d.The mean and standard deviation obtained are as follows:

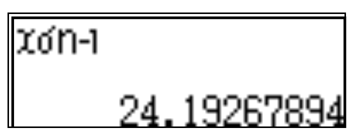
5 *(DATA D):





10 *(DATA D):





5 *(DATA D) : mean $\bar{x} = 341.71$ and standard deviation $s = 12.096$
10* (DATA D): mean $\bar{x} = 683.42$ and standard deviation $s = 24.192$.

e. If all measurements in a data set are multiplied by a fixed number d, then the mean of the new measurements is d multiplied by $\bar{x}$, the original mean. Similarly, the deviations ($x - \bar{x}$) get multiplied by d. Hence, it follows that, s^2 gets multiplied by d^2, and s by $|d|$. Note that the standard deviation is never negative.

EXERCISES

Consider the following exercises on the standard deviation:

1. The following data are the weights (in pounds) of eight 40 year old women:

Women	A	B	C	D	E	F	G	H
Weight	125	125	125	125	125	125	125	125

Calculate the standard deviation and give an explanation for the resulting value.

Solution:

If we calculate the variance and the standard deviation for these data, their values are zero. This is because there is no variation in the values of this data set.

2. Consider the following data sets:

Data Set A	10	23	35	6	38
Data Set B	17	30	42	13	45

a. How do the values of Data Set B differ from the values of Data Set A?
b. Calculate the standard deviation for each of these two data sets. Comment on the relationship between the two standard deviations.

Solution:

a. Note that each value in Data Set B is obtained by adding 10 to the corresponding value of Data Set A.
b. The standard deviation obtained from both Data Sets A and B is $s = 14.363$. The spread of the values around the mean will be the same since the corresponding values in the data sets differ by the same constant.

3. Consider the following application:

On a given day, the noon temperature measurements in Fahrenheit (^0F) reported by 14 weather stations were

A	B	C	D	E	F	G	H	I	J	K	L	M	N	**Station**
76	82	75	78	78	75	74	77	79	77	81	74	80	79	**Temp**

a. Find the mean temperature in ^0F.

b. The Celsius scale (^0C) is related to Fahrenheit (^0F) scale by ($C = \frac{5}{9}(F - 32)$). What is the mean temperature in ^0C?

Solution:

a. The mean temperature in ${}^{0}F$ is given by $\bar{x} = 77.5^{0}F$.

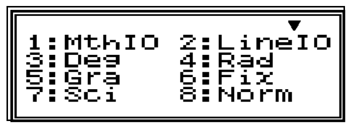
b. Using the results in the exploratory activity given earlier, the mean temperature in ${}^{0}C$ is calculated by:

$$C = \frac{5}{9}(\bar{x} - 32) = 25.28^{0}C.$$

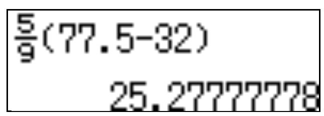
To calculate b) we follow the following calculator keystrokes:

[Operations]

- Press Mode. Select 1: COMP
- Press Shift Setup



- Select MthIO
- Enter quotient sign
- Enter 5
- Enter 9
- Enter (77.5-32)



4. Consider the following $n = 7$ observations:

27	28	32	34	37	28	33

a. What is the mean for this set?

b. Change the value $x = 27$ to $x = 25$. What is the mean for the new data set?

c. Change the value $x = 27$ to $x = 12$ then $x = 3$. As the value of x gets smaller, What do you observe about the mean?

Solution:

a. We obtain the mean $\bar{x} = 31.285$.

b. The mean is $\bar{x} = 31$.

c. If we change $x = 27$ to $x = 12$, we obtain $\bar{x} = 29.142$. If we change $x = 27$ to $x = 3$, we obtain a mean value $\bar{x} = 27.857$. As the value of x gets smaller, the sample mean gets smaller.

The sample mean gets affected more than other measures with the presence of an extreme value in the data.

Random Sampling

Random sampling is one of the basic sampling methods that can be used to obtain a representative sample. In order to obtain a random sample, each subject of the population must have an equal chance of being selected.

The best way to obtain a random sample is to use a list of random numbers. In this note, we present ways of how to obtain a random sampling with the aid of a scientific calculator. When random numbers are generated be a calculator, there is a better chance that every number has an equally likely chance of getting selected, unlike for instance when numbers are placed in a hat or mixed, one can never be sure that they are thoroughly mixed and selected so that each number has an equal chance of getting selected.

ACTIVITY

A population consists of $N = 999$ experimental units. Use the calculator to select a sample of $n = 25$ experimental units. What is the probability that each experimental unit is selected for inclusion in the sample?

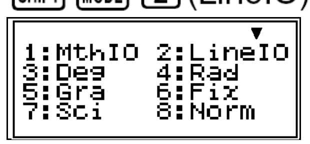
Solution:

For this situation, we are going to use three digit numbers as random numbers. We assign a three digit number to each of the sampling units.

The scientific calculator has a random number generation capability, **Ran#** which can be used to generate the random numbers. The command generates a random number larger than zero and less than one. Eventually, the random numbers produced are spread evenly over the whole interval from zero to one.
The following calculator commands will be used to generate random numbers:

[Operations]

- Press Mode. Select 1: COMP — [MODE] [1]
- Press SHIFT SET UP
- Select 2:LineIO — [SHIFT] [MODE] [2] (LineIO)



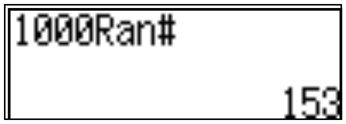
(Note: We will use the same mode and select 2:LineIO throughout the whole paper.)

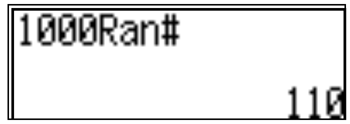
- Enter 1000Ran# to generate a random number between 0 and 1000

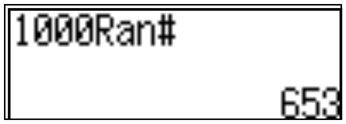
[1] [0] [0] [0] [SHIFT] [•] [=]

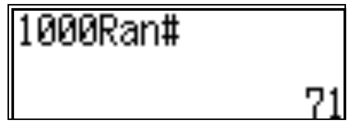
After the Ran# command is executed with [=] key, a new random number will be produced with every press of the [=] key.

The screens below show a sequence of four random numbers generated by the calculator. By their nature, random numbers are unpredictable, so the learner may not obtain the same four numbers as shown here when he tries this activity. He may obtain a different set of four numbers.









The probability that each experimental unit is selected for inclusion in the sample is $\frac{1}{999} = 0.001$.

EXERCISES

1. A political analyst wishes to select a sample of $n = 40$ from a list of voters in the town's computer database, where the computer listing involves voters who are assigned voter numbers 1 to 6000. Use the calculator to identify the voters to be included in the sample.

Solution:

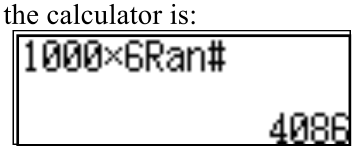
To choose the voter included in the sample, we generate a random number from the calculator.
The command **6Ran#** generates a number between 0 and 6. To generate a number from 0 to 6000 we use the following calculator command:

[Operations]

- Enter 1000 6Ran# to generate a random number between 0 and 6000

[1] [0] [0] [0] [×] [6] [SHIFT] [•] [=]

A sample random number generated by the calculator is:



2. A researcher, who is a chemist is testing a new method for measuring the amount of titanium(Ti) in ore samples. She chooses 10 ore samples of the same weight for her experiment. Five of the samples will be measured using a standard method, and the other 5 using the new method. Use random numbers to assign the 10 ore samples to the new and standard groups.

Solution:

The researcher selects what she believes are 10 representative ore samples and proceeds to randomly select the five samples to be measured with each method.
Assign the numbers 1 to 10 to the samples. The five samples selected for the new method may correspond to five one-digit random numbers.
We generate three digit numbers using the calculator. We have the following calculator keystrokes:

[Operations]

- Enter 1000Ran# to generate a random number between 0 and 1000

[1] [0] [0] [0] [SHIFT] [•] [=]

For instance, we have the numbers as follows:

948 247 913 614 608

Observe that we cannot select the same ore sample twice, so we skip any digit that has already been chosen.
As we go from left to right in the table, we select 9, then 4, 8, 2. We skip the next 4 then select 7. The ore samples 9,4,8,2 and 7 will be measured using the new method. The other samples, which are 1,3,5,6 and 10 will be measured using the standard method.

3. A computer database at a university contains files of $N = 400{,}000$ consisting of its graduates. The university wants to select $n = 4{,}000$ files for review. Select a simple random sample of $4{,}000$ files from this database.

Solution:

1. Label each graduate's file with a six-digit number. For instance, a graduate will have file number 312457.
2. To select a file number to be included in the sample, generate two three digit random numbers using the calculator. For instance, on the first try you get 120 and on the second try you obtain 27. Then the six digit number will be 120027.

[Operations]

- Enter 1000Ran# to generate the first random number between 0 and 1000

  [1] [0] [0] [0] [SHIFT] [•] [=]

- We enter 1000Ran# again to generate the second random number between 0 and 1000

  [1] [0] [0] [0] [SHIFT] [•] [=]

3. Suppose if you get 572 on the first try and say 120 on the next. Since there is no student corresponding to 572120, then ignore this number and obtain another by the same means and so on. We carry on until the required number of observations, or sample size is carried on.

EXPLORATORY TASKS

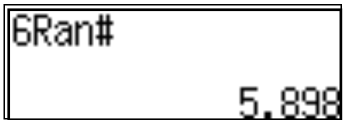
Here are some questions on how to generate random numbers using the calculator. Students can check and explore using the calculator.

1. How would one generate a random number larger than zero and less than 4? How about a random number larger than zero and less than 7, and so on?

Solution:

Observe that the calculator command **Ran#** generates a random number r larger than zero and less than one:
Ran# = r, $0 < r < 1$
From Exercise 1), to generate a random number larger than zero and less that 6, we use **6Ran#:**
6Ran# $= 6r$, $0 < 6r < 6$

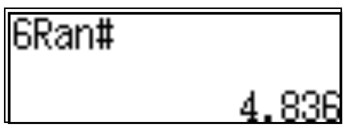
Thus, we use: **4Ran#** $= 4r$, $0 < 4r < 4$
7Ran# $= 7r$, $0 < 7r < 7$
and so on.

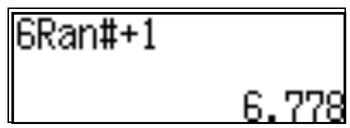


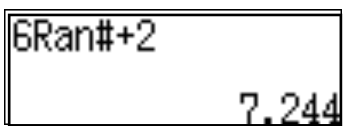
2. How would one generate a random number between 1 and 7? How about a random number between 2 and 8? 3 and 9, and so on?

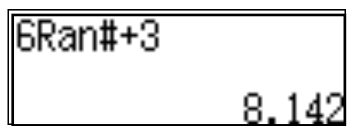
Solution:

From Exercise 1, **6Ran#** $= 6r$, $0 < 6r < 6$
This implies **6Ran# +1** $= 6r + 1$, $1 < 6r + 1 < 7$
Also, **6Ran# +2** $= 6r + 2$, $2 < 6r + 2 < 8$
6Ran# +3 $= 6r + 3$, $3 < 6r + 3 < 9$
and so on.



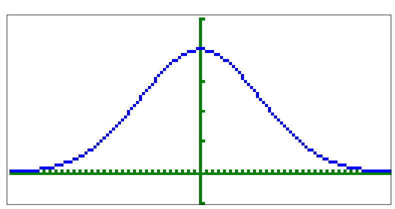






Normal Probability Distribution

The normal distribution is one of the many probability distributions that a continuous random variable can possess. The normal distribution is the most important and most widely used of all probability distributions. A large number of phenomena in the real world are normally distributed exactly or approximately.
The normal probability distribution or the normal curve is a bell-shaped (symmetric) curve. Its mean is denoted by μ and its standard deviation by σ. The normal distribution with $\mu = 0$ and $\sigma = 1$ is called a standard normal distribution. A continuous random variable z that has a normal distribution is called a normal random variable.



Mean = μ

Normal distribution with mean μ and standard deviation σ.

ACTIVITIES

In this activity, we explore some properties of the normal probability distribution using a standard normal curve:

1. Find the following areas under the standard normal curve:
a. area to the left of $z = 0 = P(z < 0)$
area to the right of $z = 0 = P(z > 0)$
b. area to the left of $z = -1.2 = P(z < -1.2)$
area to the right of $z = -1.2 = P(z > -1.2)$
What is the sum of the areas obtained in questions a and b? What does this indicate?

2. Find the following areas under the standard normal curve:
a. area to the left of $z = -2.17 = P(z < -2.17)$
b. area to the right of $z = 2.17 = P(z > 2.17)$
What do you observe regarding the areas you obtained? What does this indicate?

Solution:

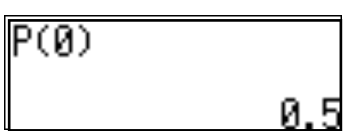
1. a. We will solve the activity with the aid of the scientific calculator. The calculator keystrokes are outlined below.

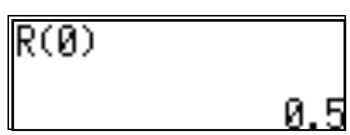
[Operations]

- Press Mode. Select 3: STAT — [MODE] [3] [AC]

We use STAT mode throughout this section

- Select 7: Distribution — [SHIFT] [1] [5]
- Select lower probability of standard normal distribution $P(z < t) = P(t)$ — [1]
- Enter 0 — [0] [=]
- Press AC — [AC]
- Select 7: Distribution — [SHIFT] [1] [5]
- Select upper probability of standard normal distribution $P(z > t) = R(t)$ — [3]
- Enter 0 — [=]





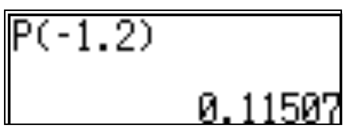
We obtain the area to the left of $z = 0 = P(z < 0) = 0.50$.

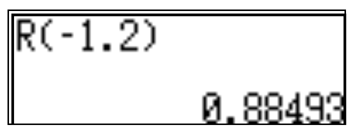
In the same manner the area to the right of $z = 0 = P(z > 0) = 0.50$.
This indicates that the area on each side of the mean(in this case is $\mu = 0$) is 0.50. One half of the total area under a normal distribution curve lies on the left side of the mean, and one half lies on the right side of the mean.

b.
[Operations]

- Select 7: Distribution [SHIFT] [1] [5]
- Select lower probability of standard normal distribution $P(z < t) = P(t)$ [1]
- Enter -1.2 [(−)] [1] [•] [2] [=]
- Press AC [AC]
- Select 7: Distribution [SHIFT] [1] [5]
- Select upper probability of standard normal distribution $P(z > t) = R(t)$ [3]
- Enter -1.2 [(−)] [1] [•] [2] [=]

Thus, area to the left of $z = -1.2 = P(z < -1.2) = 0.11507$
Also, the area to the right of $z = -1.2 = P(z > -1.2) = 0.88493$.

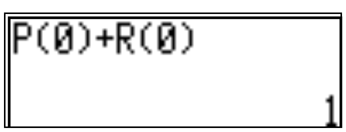


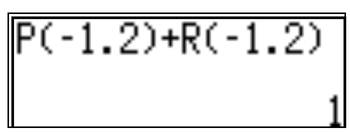


If we add the areas obtained in (a) we have:
$P(z < 0) + P(z > 0) = \text{total area} = 1.0$

Similarly, if we add the areas obtained in (b) we obtain:
$P(z < -1.2) + P(z > -1.2) = 0.11507 + 0.88493 = 1.0$

This is indicative of the property that the total area under a normal distribution curve is 1.0 or 100%.

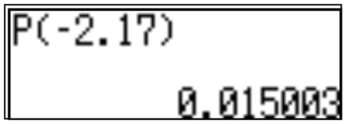


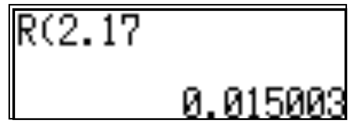


2.
[Operations]

- Select 7: Distribution [SHIFT] [1] [5]
- Select lower probability of standard normal distribution $P(z < t) = P(t)$ [1]
- Enter -2.17 [(−)] [2] [•] [1] [7] [=]
- Press AC [AC]
- Select 7: Distribution [SHIFT] [1] [5]
- Select upper probability of standard normal distribution $P(z > t) = R(t)$ [3]

Enter -2.17 [2] [•] [1] [7] [=]





The area to the left of $z = -2.17 = P(z < -2.17) = 0.015003$
The area to the right of $z = 2.17 = P(z > 2.17) = 0.015003$

The areas are equal. This is because the normal distribution curve is symmetric about the mean, which in this case is zero.

EXERCISES

Follow up Exercises to Exploratory Activity:

1. Find the area under the standard normal curve from $z = -3.5$ to $z = 0$.

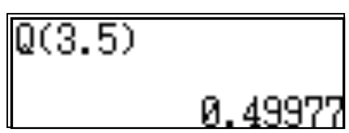
Solution:

From the exploratory activity, we know that the normal distribution is symmetric about the mean. Thus, area from $z = -3.5$ to $z = 0$ is the same as the area from $z = 0$ to $z = 3.5$.
Using the calculator, we obtain
Area from $z = -3.5$ to $z = 0$ = Area from $z = 0$ to $z = 3.5 = P(0 < z < 3.5) = 0.49977$.

The operations are given as follows:

[Operations]

- Select 7: Distribution [SHIFT] [1] [5]
- Select standard normal distribution from $z = 0$ to $z = t$. $P(0 < z < t) = Q(t)$ [2]
- Enter 3.5 [3] [•] [5] [=]



2. Find the area under the standard normal curve between $z = -1.4$ to $z = 2.7$.

Solution:

We compute the area between $z = -1.4$ to $z = 2.7$ by getting the sum of the area between $z = -1.4$ to $z = 0$ and the area between $z = 0$ to $z = 2.7$.
Just as in exercise 1, we use the notion of symmetry about the mean to conclude that the area between $z = -1.4$ to $z = 0$ is equal to

Area between $z = 0$ to $z = 1.4$ $= P(0 < z < 1.4) = 0.41924$.
The area between $z = 0$ to $z = 2.7 = P(0 < z < 2.7) = 0.49653$.
Thus,
Area between $z = -1.4$ to $z = 2.7$ is $0.41924 + 0.49653 = 0.91577$.

The calculator operations to be used are:

[Operations]

- Select 7: Distribution [SHIFT] [1] [5]
- Select standard normal distribution from $z = 0$ to $z = t$. $P(0 < z < t) = Q(t)$ [2]
- Enter 1.4 [1] [•] [4] [=]
- Press AC [AC]
- Select 7: Distribution [SHIFT] [1] [7]
- Select standard normal distribution from $z = 0$ to $z = t$. $P(0 < z < t) = Q(t)$ [2]
- Enter 2.7 [2] [•] [7] [=]

Q(1.4)	Q(2.7)	Q(1.4)+Q(2.7)
0.41924	0.49653	0.91577

3. Find the area under the standard normal curve between $z = 1.49$ to $z = 2.21$.

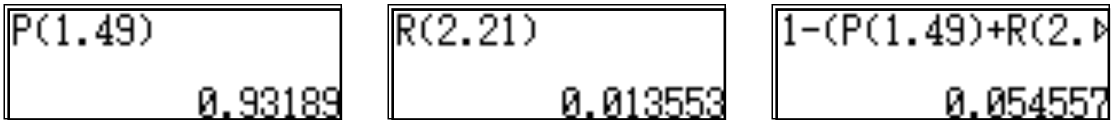
Solution:

We first compute the area to the left of $z = 1.49$ and area to the right of $z = 2.21$:
Area to the left of $z = 1.49 = P(z < 1.49) =$ 0.93189
Area to the right of $z = 2.21 = P(z > 2.21) =$ 0.013553.
Now, from the exploratory activity, we know that the total area under the normal curve is equal to 1 Therefore
Area between $z = 1.49$ to $z = 2.21$ = $P(1.49 < z < 2.21)$ = $\{1\text{-}[P(z < 1.49) + P(z > 2.21)]\}$
$= 1\text{-}(0.93189 + 0.013553) = 0.054557$.

The calculator keystrokes are:

[Operations]

- Select 7: Distribution [SHIFT] [1] [5]
- Select lower probability of standard normal distribution $P(z < t) = P(t)$ [1]
- Enter 1.49 [1] [•] [4] [9] [=]
- Press AC [AC]
- Select 7: Distribution [SHIFT] [1] [5]
- Select upper probability of standard normal distribution $P(z > t) = R(t)$ [3]
- Enter 2.21 [2] [•] [2] [1]

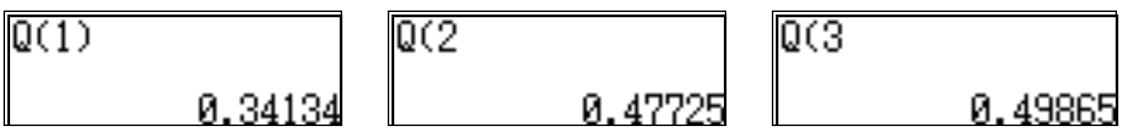


4. For a standard normal distribution, find the following areas:

a. area within one standard deviation of the mean
b.area within two standard deviations of the mean
c. area within three standard deviations of the mean

Solution:

a. First, we calculate the area between $z = 0$ of $z = 1.0$ as follows:
$P(0 < z < 1.0) = 0.34134$
Then the area within one standard deviation of the mean using symmetry is $P(-1.0 < z < 1.0) =$ $2[P(0 < z < 1.0)] = 0.68268$.
The solutions for (b) and (c) follow similarly:
b. $P(0 < z < 2.0) = 0.47725$
This implies the area within two standard deviations of the mean is
$P(-2.0 < z < 2.0) = 2[P(-2.0 < z < 2.0)] = 0.9545$.
c. $P(0 < z < 3.0) = 0.49865$
Thus the area within three standard deviations of the mean is
$P(-3.0 < z < 3.0) = 2[P(-3.0 < z < 3.0)] = 0.9973$.



The calculator keystrokes are given below:

[Operations]

- Select 7: Distribution [SHIFT] [1] [5]
- Select standard normal distribution from $z = 0$ to $z = t$. $P(0 < z < t) = Q(t)$ [2]
- Enter 1 [1] [=]
- Press AC [AC]
- Select 7: Distribution [SHIFT] [1] [5]
- Select standard normal distribution from $z = 0$ to $z = t$. $P(0 < z < t) = Q(t)$ [2]
- Enter 2 [2] [=]
- Select standard normal distribution from $z = 0$ to $z = t$. $P(0 < z < t) = Q(t)$ [2]
- Enter 3 [3] [=]

Note that the above exercise is an important property of a normal distribution: the area under a normal curve that lie within one standard deviation of the mean is approximately 68%; within two standard deviations, about 95% and within three standard deviations, about 99%.

Additional Exploratory Activity:

For a standard normal distribution, find the following areas:

a. area between $z = 0$ to $z = 3.09$
b. area between $z = 0$ to $z = 4$
c. area between $z = 0$ to $z = 5$
d. area between $z = 0$ to $z = 6$

What do you observe with the areas obtained? Why do you think this is the case?

Solution:

a. $P(0 < z < 3.09) = 0.499$
b. $P(0 < z < 4) = 0.49997$
c. $P(0 < z < 5) = 0.5$
d. $P(0 < z < 6) = 0.5$

Notice that from Exercise 4 c, the total area within 3 standard deviations of the mean is 99.7%. That is why the area between $z = 0$ to $z = a$ where $a > 3$ is about 0.499 and as a increases is approximated by 0.5. The same will be true for area between $z = 0$ to $z = -a$ where $a > 3$.(see Exercise 1)

Real World Applications of Normal Probability Distribution

In real world applications, a continuous random variable may have a normal distribution with values of the mean μ and standard deviation σ that are different from 0 and 1 respectively. In such cases, the first thing to do is to convert the given normal distribution to a standard normal distribution.

The units of a normal distribution x can be converted to its corresponding units of the standard normal distribution z using the formula

$$z = \frac{x - \mu}{\sigma}$$

where μ and σ are the mean and standard deviation of the normal distribution of x respectively.

ACTIVITY

The U.S. Bureau of Statistics conducts periodic surveys to collect information on the labor market. According to one such survey, the average earnings of workers in retail trade were \$10 per hour in September 2002. Assuming that the hourly earnings of these workers in September 2002 follow a normal distribution with mean \$10 and a standard deviation of \$1.10, find the probability that the hourly earnings of a randomly selected retail trade worker in September 2002 were

a. more than \$14
b. between \$9.50 and \$10.70

Solution:

Let x be the hourly earnings of the workers. Then x has a normal distribution with mean μ =10 and standard deviation σ =1.10.

We will solve the activity with the aid of the scientific calculator.

a. To solve $P(x > 14)$, we first convert $x = 14$ to z as follows:

$$z = \frac{x - \mu}{\sigma} = \frac{14 - 10}{1.10} = 3.\overline{6363}$$

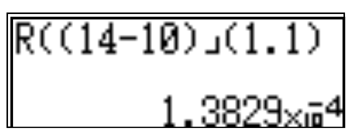
Then $P(x > 14) = P(z > 3.\overline{6363}) = 0.00013829$

There was very little probability that the hourly earnings exceeded \$14 at that time for a retail trade worker.

We use the following calculator keystrokes:

[Operations]

- Press Mode. Select 3: STAT [MODE] [3] [AC]
  (Note: We will use the same mode throughout the whole paper.)
- Select 7: Distribution [SHIFT] [1] [5]
- Select upper probability of standard normal distribution $P(z > t) = R(t)$ [3]
- Enter (14-10) [(] [1] [4] [−] [1] [0] [)]
- Enter quotient [÷]
- Enter (1.10) [(] [1] [•] [1] [)] [=]



b. To calculate $P(9.50 < z < 10.70)$, we convert x values to z:

For $x = 9.50$: $z = \dfrac{9.50 - 10}{1.10} = -0.\overline{45}$

For $x = 10.70$: $z = \dfrac{10.70 - 10}{1.10} = 0.\overline{63}$

Then $P(9.50 < z < 10.70) = P(-0.\overline{45} < z < 0.\overline{63})$

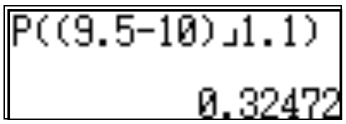
$= \{1\text{-}[P(z < -0.\overline{45}) + P(z > 0.\overline{63})]\} = \{1\text{-}[0.32472 + 0.26227)]\} = 0.41301$

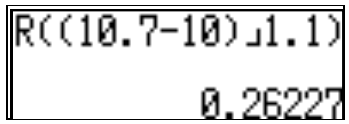
The probability that the hourly earnings of a retail trade worker was between \$9.50 and \$10.70 is about 41.3%.

We use the following calculator commands:

[Operations]

- Select 7: Distribution — SHIFT 1 5
- Select lower probability of standard normal distribution $P(z < t) = P(t)$ — 1
- Enter (9.5-10) — (9 • 5 − 1 0)
- Enter quotient — ⌸
- Enter (1.10) — (1 • 1) =
- Select upper probability of standard normal distribution $P(z > t) = R(t)$ — 3
- Enter (10.70-10) — (1 0 • 7 − 1 0)
- Enter quotient — ⌸
- Enter (1.10) — (1 • 1) =





Alternative Solution:

We provide an alternative solution using calculus:
For a normal distribution, we use the probability density function given by

$$f(x) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} dx$$

where μ and σ are the mean and standard deviation.

a. To solve $P(x > 14)$ we use the integral

$$P(x > 14) = \int_{14}^{\infty} \frac{1}{1.10\sqrt{2\pi}} e^{-(x-10)^2/(2(1.1)^2} dx$$

To avoid the improper integral we could approximate it by the integral from 14 to 30(its quite safe to say that the hourly earnings will not exceed this much).
We obtain

$$P(x > 14) = \int_{14}^{30} \frac{1}{1.10\sqrt{2\pi}} e^{-(x-10)^2/(2(1.1)^2} dx = 0.000138256$$

We get almost the same approximation as obtained in the first solution.

b. To calculate $P(9.50 < z < 10.70)$ we use the formula

$$P(9.50 < z < 10.70) = \int_{9.5}^{10.7} \frac{1}{1.10\sqrt{2\pi}} e^{-(x-10)^2/(2(1.1)^2} dx$$

We have $P(9.50 < z < 10.70) = 0.413012$, the same answer we obtained using the first solution.

The following calculator keystrokes will be used for this part of the chapter:

[Operations]

- Press Mode. Select 1: COMP [MODE] [1]
- Select integral sign [∫□]
- Enter the function, then the upper limit and lower limit of the integral.

Navigate using the cursor keys ▼ ▲

EXERCISE

The following exercise allows the student to explore and compare normal distributions with different means and standard deviations.

Suppose the distribution of incomes in the following countries is normal:

Country	US	Canada	Switzerland	Germany	Sweden
Mean Household Income ($)	38,000	35,000	39,000	34,000	32,000
Standard Deviation ($)	21,000	17,000	16,000	14,000	11,000

Source: Luxembourg Income Study, New York Times, August 1995

a. Which country has a higher proportion of poor families- with household income of $12,000 or less?
b. Which country has a higher proportion of wealthy families – with household income of at least $100,000?

Solution:

Let x be the household income.
a. We find $P(x \le 12{,}000) = P(x < 12{,}000)$ for each country; converting the variable x to z in each case.

U.S. : $P(z < \dfrac{12{,}000 - 38{,}000}{21{,}000}) = P(z < -1.238) = 0.1078$

Canada: $P(z < \dfrac{12{,}000 - 35{,}000}{17{,}000}) = P(z < -1.3529) = 0.088$

Switzerland: $P(z < \dfrac{12{,}000 - 39{,}000}{16{,}000}) = P(z < -1.6875) = 0.0458$

Germany: $P(z < \dfrac{12{,}000 - 34{,}000}{14{,}000}) = P(z < -1.5714) = 0.058$

Sweden: $P(z < \dfrac{12{,}000 - 32{,}000}{11{,}000}) = P(z < -1.\overline{81}) = 0.0345$

From the results above, we can conclude that the United States has the highest proportion of poor families- 10.78%

We illustrate the following calculator keystrokes for $P(z < \frac{12{,}000 - 38{,}000}{21{,}000})$. The rest may be done similarly.

[Operations]

- Press Mode. Select 3: STAT — [MODE] [3] [AC]
- Select 7: Distribution — [SHIFT] [1] [5]
- Select lower probability of standard normal distribution $P(z < t) = P(t)$ — [1]
- Enter (12,000-38,000) — [(] [1] [2] [0] [0] [0] [−] [3] [8] [0] [0] [0] [)]
- Enter quotient — [÷]
- Enter (21,000) — [(] [2] [1] [0] [0] [0] [)] [=]

```
P((12000-38000)▸
          0.10784
```

b. We now find $P(x \geq 100{,}000) = P(x > 100{,}000)$ for each country.

U.S. : $P(z > \frac{100{,}000 - 38{,}000}{21{,}000}) = P(z > 2.9524) = 0.0015767$

Canada: $P(z > \frac{100{,}000 - 35{,}000}{17{,}000}) = P(z > 3.8235) = 0.0000658$

Switzerland: $P(z > \frac{100{,}000 - 39{,}000}{16{,}000}) = P(z > 3.8125) = 0.0000688$

Germany: $P(z > \frac{100{,}000 - 34{,}000}{14{,}000}) = P(z > 4.7142) = 0.000001214$

Sweden: $P(z > \frac{100{,}000 - 32{,}000}{11{,}000}) = P(z > 6.\overline{18}) = 0.000000000318$

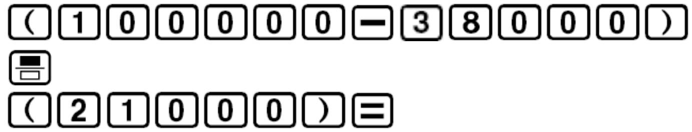
[Note: For a continuous random variable, $P(x \leq a) = P(x < a)$ or $P(x \geq a) = P(x > a)$]
Thus from our results, we can conclude that the United States has the highest proportion of rich families- 0.16%

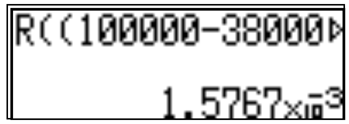
We illustrate the following calculator keystrokes for $P(z > \frac{100{,}000 - 38{,}000}{21{,}000})$. The rest may be done similarly.

[Operations]

- Press Mode. Select 3: STAT — [MODE] [3] [AC]
- Select 7: Distribution — [SHIFT] [1] [5]
- Select upper probability of standard normal distribution $P(z > t) = R(t)$ — [3]

- Enter (100,000-38,000)
- Enter quotient
- Enter (21,000)





An observation we can make from the results obtained in exercise b above is that the probabilities obtained are very, very small or geometrically the areas under the normal curve is almost zero. This is because the normal distribution curve, though it never meets the horizontal axis, beyond the points represented by $\mu - 3\sigma$ and $\mu + 3\sigma$ becomes so close to the x axis that the area under the curve beyond these points in both directions can be taken as virtually zero.

Confidence Intervals

A confidence interval is an interval constructed around the value of a sample statistic to estimate the corresponding population parameter. The confidence level, denoted by $(1-\alpha)100\%$, states how much confidence we have that a confidence interval contains the true population parameter.

The $(1-\alpha)100\%$ confidence interval for μ for a large sample n, $(n \geq 30)$ is

$$\bar{x} \pm z\sigma_{\bar{x}} \text{ if } \sigma \text{ is known}$$

$$\bar{x} \pm zs_{\bar{x}} \text{ if } \sigma \text{ is not known}$$

$$\text{where } \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} \text{ and } s_x = \frac{s}{\sqrt{n}}$$

The standard deviation of $\bar{x}$ is $\sigma_{\bar{x}}$, the population standard deviation is σ and the sample standard deviation is s. The value of z used here is assumed from the standard normal distribution for the given confidence level.

The calculation of confidence intervals becomes easier with the aid of a scientific calculator.

ACTIVITY

Answer the following problem:

For a population, the value of the standard deviation is 2.65. A sample of 35 observations taken from this population produced the following data:

42	51	42	31	28	36	49
29	46	37	32	27	33	41
47	41	28	46	34	39	48
26	35	37	38	46	48	39

29	31	44	41	37	38	46

a. What is the maximum error of estimate for μ?
b. Construct various 95%, 98% and 99% confidence intervals for μ.
c. What do you observe regarding these intervals as the confidence level becomes greater?

Solution:
a. The maximum error of estimate $z\sigma_{\bar{x}}$ for μ is the quantity that is subtracted from and added to the value of $\bar{x}$ to obtain a confidence interval for μ. Note that for the above example, since the population standard deviation σ is known and given, we can calculate $\sigma_{\bar{x}}$ as follows:

$$\sigma_{\bar{x}} = \frac{2.65}{\sqrt{35}} = 0.447931755$$

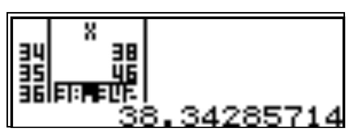
This implies the maximum error estimate for μ, $z\sigma_{\bar{x}} = (2.33)(0.447931755) \approx 1.04$ (See * below for the explanation on the z values used)

To be able to solve questions b and c, we first calculate the mean for the given population using the following calculator commands:

[Operations]

- Press Mode. Select 3: STAT [MODE] [3]
- Select 1: 1-VAR [1]
- Enter each of the data in the table then press [AC]
- Calculate the mean $\bar{x}$ [SHIFT] [1] [4] [2] [=]
- Whenever we want to refer to the same data again [SHIFT] [1] [2]
  The data will be kept in the calculator's memory.

We obtain the mean of the given population to be $\bar{x} = 38.342$.



To determine the confidence intervals, we use the following values for z :

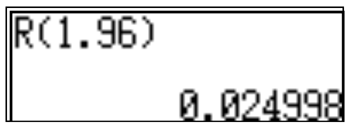
Confidence Coefficient (1-α)	$\frac{\alpha}{2}$	z
0.95	0.025	1.96
0.98	0.01	2.33
0.99	0.005	2.58

The value z has an area of $\frac{\alpha}{2}$ under the right tail of a standard normal distribution curve. For instance, the area from $z = 1.96$ to ∞ is about 0.025, the area from $z = 2.33$ to ∞ is about 0.01 and the area from $z = 2.58$ to ∞ is about 0.005.

We can verify the results $P(z > 1.96)$, $P(z > 2.33)$ and $P(z > 2.58)$ through the calculator using the commands below. We illustrate how to obtain $P(z > 1.96)$:

[Operations]

- Select 7: Distribution [SHIFT] [1] [5]
- Select upper probability of standard normal distribution $P(z > t) = R(t)$ [3]
- Enter, 1.96) [1] [•] [9] [6] [)] [=]



Using the values for z, we now have the following calculations pertaining to the confidence intervals:

Confidence Coefficient (1-α)	z	$\bar{x} - z\sigma_{\bar{x}}$	$\bar{x} + z\sigma_{\bar{x}}$	**Confidence Interval**
0.95	1.96	37.464	39.22	$[37.464, 39.22]$
0.98	2.33	37.298	39.386	$[37.298, 39.386]$
0.99	2.58	37.186	39.497	[37.186,39.497]

The confidence intervals obtained have widths $1.756, 2.088, 2.311$ corresponding to confidence levels 95%, 98% and 99% respectively. This is indicative of the fact that for greater confidence, we would have to move further away from the mean, so the factor z would increase and so would the maximum error estimate. This means the confidence intervals will become wider.

EXERCISES

Solve the following:

A hospital administration wants to estimate the mean time spent by patients waiting for treatment at the emergency room. The waiting times (in minutes) recorded for a random sample of 32 such patients are given below.

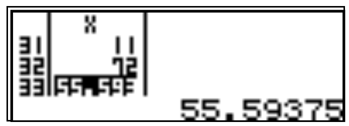
110	42	88	19	35	76	10	151
2	44	27	77	53	102	66	39

20	108	92	55	14	52	3	62
78	15	60	121	40	35	11	72

a. What is the point estimate of the mean waiting time for all patients at this emergency room?
b. Construct a 98% confidence interval for the corresponding population mean.

Solution:

a. We follow the steps given in the previous activity to calculate the mean, $\bar{x}$. We obtain $\bar{x} = 55.593$. Thus the point estimate of the mean waiting time for all patients at the emergency room is about 55.59 minutes.



b. For this problem, the population standard deviation σ is not known. We use the calculator to compute the sample standard deviation s.

We use the following calculator commands to compute the sample standard deviation, s.

[Operations]
After computing the mean, we can compute the standard deviation by pressing AC first (assuming data still is in memory of the calculator) then

- Calculate the sample standard deviation $x\sigma n-1$

SHIFT 1 4 4 =

We obtain the value $s = 37.521$.

This implies the estimator $s_{\bar{x}} = \frac{s}{\sqrt{n}} = \frac{37.521}{\sqrt{32}} = 6.632838$. This estimates $\sigma_{\bar{x}}$.

Now since we are going to construct a 98% confidence interval for the corresponding population mean, we will use $z = 2.33$.

Plugging in the values to the formula $\bar{x} \pm z s_{\bar{x}}$ gives:

$$\bar{x} - z s_{\bar{x}} = 55.593 - (2.33)(6.632838) = 40.138$$

$$\bar{x} + z s_{\bar{x}} = 55.593 + (2.33)(6.632838) = 71.048$$

The confidence interval for the corresponding population mean is $[40.138, 71.048]$.

Hypothesis Testing

In a test of hypothesis, we test a certain given theory or belief about a population parameter. We find out, using sample information, whether or not a given claim about a population parameter is true.
In tests of hypotheses about μ for a large sample n, the random variable

$$z = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}} \text{ or } \frac{\bar{x} - \mu}{s_{\bar{x}}}$$

where $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$ and $s_{\bar{x}} = \frac{s}{\sqrt{n}}$

is called the test statistic. The standard deviation of $\bar{x}$ is $\sigma_{\bar{x}}$, the population standard deviation is σ and the sample standard deviation is s.The test statistic can be defined as a rule or criterion that is used to make the decision whether or not to reject the null hypothesis.

Hypothesis testing can be conducted more efficiently with the aid of a scientific calculator.

ACTIVITY

Answer the following problem:

According to a basketball coach, the mean height of all male college basketball players is 74 inches. A random sample of 25 such players produced the following data on their heights:

68	74	77	69	71	79	69
78	78	72	79	76	70	
76	83	76	67	74	85	
75	68	83	82	69	81	

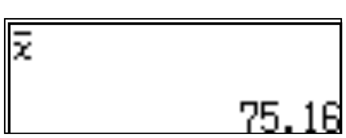
Test at the 5% significance level whether the mean height of all male college basketball players is different from 74 inches.
Assume that the heights of all male college basketball players are approximately normal distributed. What is your conclusion?

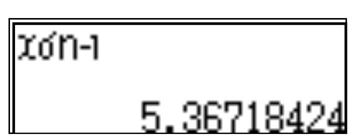
Solution:

Let μ = 74 inches be the mean height of male college basketball players and $\bar{x}$ be the corresponding mean for the sample.

We first calculate the mean $\bar{x}$ for the sample and the sample standard deviation s using the calculator

We obtain the mean to be $\bar{x} = 75.16$ and the sample standard deviation $s = 5.3672$.





We are to test whether the mean height of all male college basketball players is different from 74 inches. The significant level is α is 0.05; that is the probability of rejecting the null hypothesis H_o when it actually is true should not exceed 0.05. Since we are testing to find whether or not the mean height of all male college basketball players is different from 74 inches, we write the null and alternative hypothesis as follows:

Null hypothesis: $H_0 : \mu = 74$ (the mean height of all male college basketball players is 74 inches)

Alternative hypothesis: $H_1 : \mu \neq 74$(the mean height of all male college basketball players is different form 74 inches)

By a null hypothesis, we mean a claim about a population parameter that is assumed to be true unless it is declared false. While an alternative hypothesis is a claim about a population parameter that will be true if the null hypothesis is false.

Now, we determine the rejection and non-rejection regions. The significance level α is 0.05. The $\neq$ sign in the alternative hypothesis indicates that the test is two tailed with two rejection regions, one in each tail of the normal distribution curve of $\bar{x}$. Because the total area of both rejection regions is 0.05, the area of the rejection region in each tail is 0.025. Two critical points separate the two rejection regions from the non-rejection region. The critical points of z are 1.96 and -1.96.

We can check that $P(z > 1.96) = 0.025$ and $P(z < -1.96) = 0.025$ using the calculator. We confirm these values using the following calculator commands:

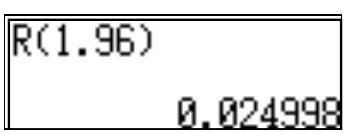
[Operations]

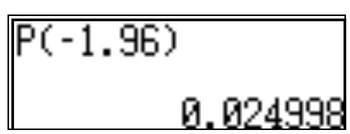
- Press Mode. Select 3: STAT [MODE] [3]
- Press AC [AC]
- Select 7: Distribution [SHIFT] [1] [5]
- Select upper probability of standard normal distribution $P(z > t) = R(t)$ [3]
- Enter 1.96) [1] [•] [9] [6] [)] [=]
- Select 7: Distribution [SHIFT] [1] [5]
- Select lower probability of standard normal distribution $P(z < t) = P(t)$ [1]
- Enter -1.96) [(−)] [1] [•] [9] [6] [)] [=]

We now have the following calculations:

$P(z > 1.96) = R(1.96) = 0.025$

$P(z < -1.96) = P(-1.96) = 0.025$





To calculate the value of the test statistic, we use the formula $z = \frac{\bar{x} - \mu}{s_{\bar{x}}}$.

Solving for $s_{\bar{x}}$, we have $s_{\bar{x}} = \frac{s}{\sqrt{n}} = \frac{5.3672}{\sqrt{25}} = 1.07344$.

Thus, $z = \dfrac{\bar{x} - \mu}{s_{\bar{x}}} = \dfrac{75.16 - 74}{1.07344} = 1.081$.

The value of the test statistic $z = 1.081$ is between -1.96 and 1.96, so we do not reject H_0.

Now, since we decided not to reject H_0 in favor of an alternative stating that the mean differed from 74 inches, we conclude that the mean is not significantly different from 74 inches at the 5% level of significance.

EXERCISES

1. Consider null and alternative hypothesis as follows:

Null hypothesis: $H_0 : \mu = 45$

Alternative hypothesis: $H_1 : \mu < 74$

a. A random sample of 100 observations produced a sample mean of 43 and a standard deviation of 5. Using $\alpha = 0.01$, would you reject the null hypothesis?

b. Another random sample of 100 observations taken from the same population produced a sample mean of 43.8 and a standard deviation of 7. Using $\alpha = 0.01$, would you reject the null hypothesis?

Solution:

a. Let mean $\bar{x} = 43$ and standard deviation $\sigma_{\bar{x}} = \dfrac{5}{\sqrt{100}} = 0.5$. Then the value of the test statistic z for $\bar{x} = 43$ is

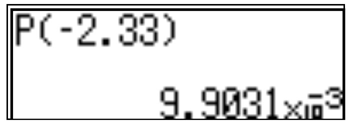
$$z = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}} = \frac{43 - 45}{0.5} = -4.$$

The significance level is 0.01. The < sign in the alternative hypothesis indicates that the test is left-tailed with the rejection region in the left tail of the sampling distribution curve of $\bar{x}$. The critical value of z is -2.33. We are assuming here that the sampling distribution of $\bar{x}$ is a normal distribution.

Using the commands below, we can check that $P(z < -2.33) = P(-2.33) = 0.01$

[Operations]

- Press Mode. Select 3: STAT — MODE 3
- Press AC — AC
- Select 7: Distribution — SHIFT 1 5
- Select lower probability of standard normal distribution $P(z < t) = P(t)$ — 1
- Enter -2.33) — (−) 2 • 3 3) =



Now, the value of the test statistic we obtained above is $z = -4$, and is less than -2.33. Consequently, we reject. Therefore, we can say that the sample mean $\bar{x} = 43$ is too far away from the hypothesized population mean $\mu = 45$.

b. Let mean $\bar{x} = 43.8$ and standard deviation $\sigma_{\bar{x}} = \frac{7}{\sqrt{100}} = 0.7$. Then the value of the test statistic z for $\bar{x} = 43.8$ is $z = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}} = \frac{43.8 - 45}{0.7} = -1.714$. The value of the test statistic $z = -1.714$ is greater than the critical value -2.33, and it falls in the non-rejection region. As a result, we fail to reject H_0. By not rejecting the null hypothesis, we are saying that the information obtained from the sample is not strong enough to reject the null hypothesis.

In this part of the discussion we present activities on how to use a scientific calculator in performing the test of hypothesis about the difference between two population means μ_1, μ_2.

In hypothesis testing about $\mu_1 - \mu_2$, there are three situations:

i. Testing an alternative hypothesis that the means of two populations are different, that is, $\mu_1 \neq \mu_2$ or $\mu_1 - \mu_2 \neq 0$

ii. Testing an alternative hypothesis that the mean of the first population is greater than the mean of the second population, $\mu_1 > \mu_2$ or $\mu_1 - \mu_2 > 0$

iii. Testing an alternative hypothesis that the mean of the first population is less than the mean of the second population, $\mu_1 < \mu_2$ or $\mu_1 - \mu_2 < 0$.

The test statistic z for $\overline{x_1} - \overline{x_2}$ is given by the formula

$$z = \frac{(\overline{x_1} - \overline{x_2}) - (\mu_1 - \mu_2)}{\sigma_{\overline{x_1} - \overline{x_2}}}$$

where $\overline{x_1}, \overline{x_2}$ are the respective means of the samples drawn from population 1 and 2; μ_1, μ_2 are the respective means of populations 1 and 2 and σ_1, σ_2 are the respective standard deviations of populations 1 and 2.

The value of $\mu_1 - \mu_2$ is substituted from the null hypothesis H_0. If the values of σ_1, σ_2 are not known, we replace the standard deviation of $\overline{x_1} - \overline{x_2}$, denoted by $\sigma_{\overline{x_1} - \overline{x_2}}$ with $s_{\overline{x_1} - \overline{x_2}} = \sqrt{\frac{s_1^{\,2}}{n_1} + \frac{s_2^{\,2}}{n_2}}$ where s_1, s_2 are the standard deviations of the two samples selected from the two populations. We assume that n_1 is the size of the sample drawn from population 1, and n_2 is the size of the sample drawn from population 2.

ACTIVITY

A random sample of 13 male college students who hold jobs gave the following data on their general point averages (GPA):

3.12	2.84	2.43	2.15	3.92	2.45	2.73
3.06	2.36	1.93	2.81	3.27	1.83	

Another random sample of 16 female college students who also hold jobs gave the following data on their GPAs.

2.76	3.84	2.24	2.81	1.79	3.89	2.96	3.77
2.36	2.81	3.29	2.08	3.11	1.69	2.84	3.02

Test at the 5% significance level whether the mean GPAs of all male and all female college students who hold jobs are different. Assume that the GPAs are normally distributed with unequal and unknown population standard deviations.

Solution:

From the information given, there are 13 male college students and 16 female college students; thus $n_1 = 13$ and $n_2 = 16$.
Let μ_1, μ_2 correspond to the mean GPAs of male and female college populations respectively.
Let $\overline{x}_1, \overline{x}_2$ correspond to the means of the respective samples of male and female college students.
The first step is to state the null and alternative hypotheses. We are to test if the two population means are different.

The two possibilities are
a. The mean GPAs of male and female college populations are not different, In other words, $\mu_1 = \mu_2$ or $\mu_1 - \mu_2 = 0$
b. The mean GPAs of male and female college populations are different, $\mu_1 \neq \mu_2$ or $\mu_1 - \mu_2 \neq 0$.
Considering these possibilities, the null and alternative hypotheses are:

$$H_0 : \mu_1 - \mu_2 = 0 \text{(the two population means are not different)}$$

$$H_1 : \mu_1 - \mu_2 \neq 0 \text{ (the two population means are different)}$$

To proceed in computing the test statistic z for $\overline{x}_1 - \overline{x}_2$, we first calculate $\overline{x}_1, \overline{x}_2$ using the calculator. First, we enter data pertaining to the males, then calculate $\overline{x}_1$, afterwards, we calculate $\overline{x}_2$. We have the following commands:

[Operations]

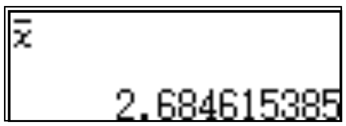
- Press Mode. Select 3: STAT [MODE] [3]

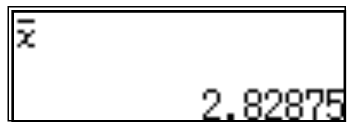
- Select 1: 1-VAR — [1]
- Select Data — [SHIFT] [1] [2]
- Enter each of the data in the table then press — [=]

Example, enter 3.12 in the first line — [3] [•] [1] [2] [=]

- Calculate the mean $\overline{x}$ — [SHIFT] [1] [4] [2] [=]
- Calculate the sample standard deviation $s\ (x\sigma n-1)$ — [SHIFT] [1] [4] [4] [=]

We obtain $\overline{x_1} = 2.685$ and $\overline{x_2} = 2.829$.



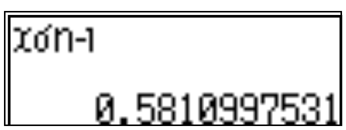


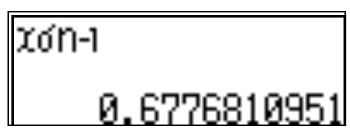
Since the values of σ_1, σ_2 are not known, we replace the standard deviation of $\overline{x_1} - \overline{x_2}$, denoted by $\sigma_{\overline{x_1}-\overline{x_2}}$ with $s_{\overline{x_1}-\overline{x_2}} = \sqrt{\frac{s_1^{\ 2}}{n_1} + \frac{s_2^{\ 2}}{n_2}}$.

We calculate s_1, s_2 as follows:
With the cursor now at the line following the last data entry,

- Calculate the sample standard deviation $s\ (x\sigma n-1)$ — [SHIFT] [1] [5] [4] [=]

We obtain $s_1 = 0.581$ and $s_2 = 0.678$.





Note that whenever data has been entered in the calculator, you can refer to it anytime by entering

[SHIFT] [1] [2]

This is useful to remember, when, for instance, after calculating the mean you may wish to calculate the standard deviation later on.

Now to determine $s_{\overline{x_1}-\overline{x_2}}$, we use $s_1 = 0.581, s_2 = 0.678$, $n_1 = 13, n_2 = 16$ in the formula:

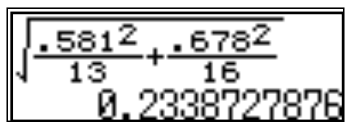
$$s_{\overline{x_1}-\overline{x_2}} = \sqrt{\frac{s_1^{\ 2}}{n_1} + \frac{s_2^{\ 2}}{n_2}}.$$

The calculator keystrokes are given below:

[Operations]

- Press Mode. Select 1: COMP — [MODE] [1]
- Select Math Mode — [SHIFT] [MODE] (SETUP) [1]
- Access square root sign — [√■]
- Access fraction symbol — [■/■]
- Enter 0.581^2 — [•] [5] [8] [1] [x^2]
- Enter13 — [▼] [1] [3]
- Enter + — [▶] [+]
- Access fraction symbol — [■/■]
- Enter 0.678^2 — [•] [6] [7] [8] [x^2]
- Enter 16 — [▼] [1] [6] [▶] [=]

We obtain $s_{\overline{x}_1-\overline{x}_2}=\sqrt{\dfrac{s_1{}^2}{n_1}+\dfrac{s_2{}^2}{n_2}}=\sqrt{\dfrac{0.581^2}{13}+\dfrac{0.678^2}{16}}=0.234$.



Now, we determine the rejection and non-rejection regions. The significance level α is 0.05. The ≠ sign in the alternative hypothesis indicates that the test is two tailed. The area in each tail of the normal distribution curve is α/2 = 0.05/2= 0.025. Two critical points separate the two rejection regions from the non-rejection region. The critical points of z are 1.96 and -1.96.

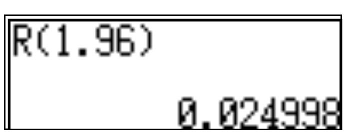
We can explore and confirm these values using the following calculator commands:

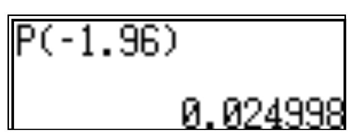
[Operations]

- Press Mode. Select 3: STAT — [MODE] [3]
- Select 7: Distribution — [SHIFT] [1] [5]
- Select upper probability of standard normal distribution $P(z > t) = R(t)$ — [3]
- Select lower probability of standard normal distribution $P(z < t) = P(t)$ — [1]

We now have the following calculations:

$$P(z > 1.96) = R(1.96) = 0.025$$
$$P(z < -1.96) = P(-1.96) = 0.025$$





To calculate the value of the test statistic, we use the formula $z=\dfrac{(\overline{x}_1-\overline{x}_2)-(\mu_1-\mu_2)}{s_{\overline{x}_1-\overline{x}_2}}$

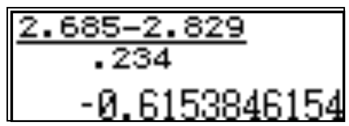
From $H_0 : \mu_1 - \mu_2 = 0$, so $z = \dfrac{(\overline{x_1} - \overline{x_2})}{s_{\overline{x_1} - \overline{x_2}}}$.

Using the calculator, the test statistic is obtained as follows:

[Operations]

- Select Math Mode — [SHIFT] [MODE] (SETUP) [1]
- Access fraction symbol — [fraction key]
- Enter $2.685 - 2.829$ — [2] [•] [6] [8] [5] [−] [2] [•] [8] [2] [9]
- Enter $s_{\overline{x_1} - \overline{x_2}} = 0.234$ — [▼] [•] [2] [3] [4] [▶] [=]

We obtain the test statistic $z = -0.615$.



Now, the value of the test statistic $z = -0.615$ is between -1.96 and 1.96, so we do not reject H_0. Now, since we decided not to reject H_0 in favor of an alternative stating that the mean GPAs of male and female college populations are different, we conclude that the mean GPAs of male and female college populations are not significantly different at the 5% level of significance.

EXERCISES

The following information is obtained from two independent samples selected from two populations:

$$n_1 = 300, \overline{x_1} = 22.0, s_1 = 4.9$$
$$n_2 = 250, \overline{x_2} = 27.6, s_2 = 4.5$$

a. Test at the 1% significant level if the two population means are different

b. Test at the 1% significant level if $\mu_1 > \mu_2$

c. Test at the 1% significant level if $\mu_1 < \mu_2$

Solution:

We first compute $s_{\overline{x_1} - \overline{x_2}} = \sqrt{\dfrac{s_1^{\ 2}}{n_1} + \dfrac{s_2^{\ 2}}{n_2}} = \sqrt{\dfrac{4.9^2}{300} + \dfrac{4.5^2}{250}} = 0.401.$

a. the null and alternative hypotheses are:

$$H_0 : \mu_1 - \mu_2 = 0 \text{(the two population means are not different)}$$

$$H_1 : \mu_1 - \mu_2 \neq 0 \text{(the two population means are different)}$$

Using the null hypothesis the test statistic $z = \dfrac{(\overline{x_1} - \overline{x_2}) - (\mu_1 - \mu_2)}{\sigma_{\overline{x_1} - \overline{x_2}}}$ becomes

$$z = \frac{(\overline{x_1} - \overline{x_2}) - (\mu_1 - \mu_2)}{\sigma_{\overline{x_1} - \overline{x_2}}} = \frac{(22 - 27.6) - 0}{0.401} = -13.965$$

The significance level is 0.01. The $\neq$ sign in the alternative hypothesis indicates that the test is two-tailed. The critical values of z for 0.005 area in each tail of the normal distribution curve are 2.33 and -2.33. Since $z = -13.965$ falls in the rejection region, we reject the null hypothesis.

b. the null and alternative hypotheses are:

$$H_0 : \mu_1 - \mu_2 = 0 \text{(the two population means are equal)}$$
$$H_1 : \mu_1 > \mu_2 \text{ (} \mu_1 \text{ is greater than } \mu_2 \text{)}$$

The > sign in the alternative hypothesis indicates that the test is right-tailed. Consequently, the critical value of z is 2.33. The test statistic $z = -13.965$ is less than 2.58, so it does not fall in the rejection region. Thus, $\mu_1 < \mu_2$.

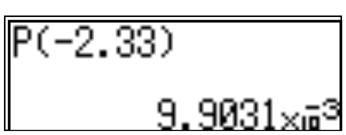
c. the null and alternative hypotheses are:

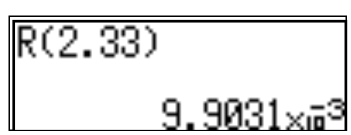
$$H_0 : \mu_1 - \mu_2 = 0 \text{(the two population means are equal)}$$
$$H_1 : \mu_1 < \mu_2 \text{ (} \mu_1 \text{ is less than } \mu_2 \text{)}$$

The < sign in the alternative hypothesis indicates that the test is left-tailed with the rejection region in the left tail of the sampling distribution curve of $\overline{x}$. The critical value of z is -2.33.
The test statistic $z = -13.965$ falls in the rejection region.
Therefore, this confirms the result in b) which means $\mu_1 < \mu_2$.
Using the commands below, we can check that $P(z < -2.33) = P(-2.33) = 0.01$ or $P(z > 2.33) = R(2.33) = 0.01$

[Operations]

- Press Mode. Select 3: STAT [MODE] [3] [AC]
- Select 7: Distribution [SHIFT] [1] [5]
- Select lower probability of standard normal distribution $P(z < t) = P(t)$ [1] [=]
- Select upper probability of standard normal distribution $P(z > t) = R(t)$ [3] [=]





EXERCISES

A company recently opened two supermarkets in two different areas. The management wants to know if the mean sales per day for these two supermarkets are different.

a. A sample of 12 days for first supermarket gives the following data on daily sales in thousand dollars:

46.3	56.78	52.33	58.29	57.69	43.22
49.1	51.37	50.12	47.32	49.22	53.79

b. A sample of 10 days for the second supermarket gives the following data on daily sales in thousand dollars:

56.33	62.34	61.35	58.29	54.78
54.40	58.44	62.33	67.92	55.64

Test at the 1% significant level whether the mean daily sales for these two supermarkets are different.

Answers:

$\overline{x_1} = 51.294, \overline{x_2} = 59.182, s_1 = 4.707, s_2 = 4.284, s_{\overline{x_1}-\overline{x_2}} = 1.919, z = -4.110$.

Reject null hypothesis, yes the mean daily sales for these two supermarkets are different.

7. Financial Mathematics

In this section we study compound interest problems using the tabular and equation capabilities of the scientific calculator.

The amount A after t years due to a principal P invested at an annual interest rate r compounded n times per year is given by

$$A = P(1+\frac{r}{n})^{nt}$$

If the interest is compounded continuously, then $A = Pe^{rt}$.

The amount A is typically referred to as the **future value** of the account while P is called the **present value**.

ACTIVITIES

1. Complete the following table that shows the value of a \$1,000 investment earning 5% interest per year after the specified time periods.

YEARS	1	2	3	4	5	6	7
VALUE							

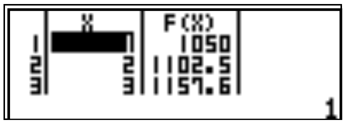
Solution:

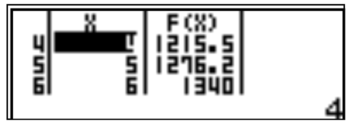
We calculate the value of the investment through the years using the formula $A = P(1+\frac{r}{n})^{nt}$ where $P = 1000$, $n = 1$ and $r = 0.05$. We use the tabular feature of the scientific calculator to solve the problem.

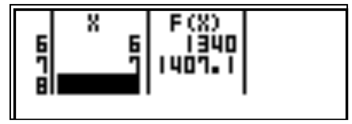
[Operations]

- Press Mode. Select 7: TABLE [MODE] [7]
  Enter the given function $1000(1.05)^x$
- Enter 1000 [1] [0] [0] [0]
- Enter $(1.05)^x$ [(] [1] [.] [0] [5] [)] [$x^{■}$] [ALPHA] [)] (X) [=]
- Enter start value, 1 [1] [=]
- Enter end value, 7 [7] [=]
- Enter step value, 1 [1] [=]

We tabulate our results as follows:







2. How long will it take money to double if it is invested at 5% interest compounded
a. annually

b. continuously

Solution:

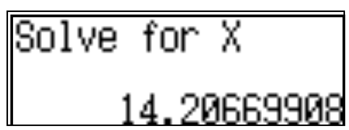
a.We solve the equation $2P = P(1.05)^x$. This implies $2 = (1.05)^x$ or equivalently, $(1.05)^x - 2 = 0$.

We use the following calculator keystrokes:

[Operations]

- Press Mode. Select 1: COMP — [MODE] [1]
- Enter $(1.05)^x$ — [(] [1] [•] [0] [5] [)] [$x^{\blacksquare}$] [ALPHA] [)] (X) [▶]
- Enter -2 — [−] [2]
- Enter = 0 — [ALPHA] [CALC] (=) [0]
- Press solve button — [SHIFT] [CALC] (SOLVE) [=]

We obtain the answer $x = 14.2$ years.



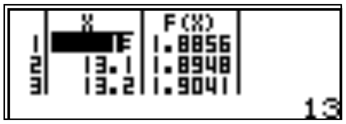
Using the tabular function of the calculator, we can verify this result by generating the future values of the money from 13 years to 14.5 years as follows:

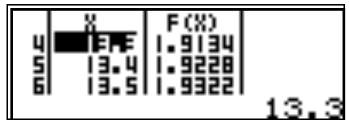
[Operations]

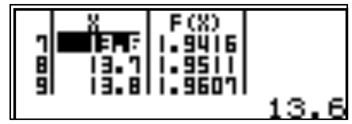
- Press Mode. Select 7: TABLE — [MODE] [7]

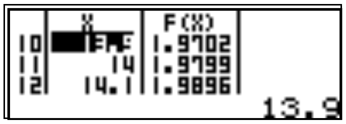
  Enter the given function $(1.05)^x$
- Enter $(1.05)^x$ — [(] [1] [•] [0] [5] [)] [$x^{\blacksquare}$] [ALPHA] [)] (X) [=]
- Enter start value, 13 — [1] [3] [=]
- Enter end value,14.5 — [1] [5] [=]
- Enter step value, 0.1 — [•] [1] [=]

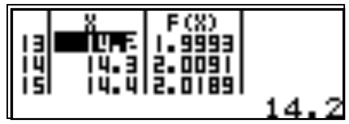
We obtain the following table:

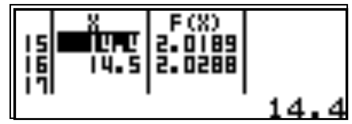






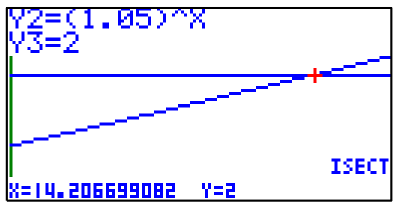






Note that when x =14.2, money is about 1.999 or approximately 2. It will take approximately 14.2 years for money to double if it is invested at 5% interest compounded annually.

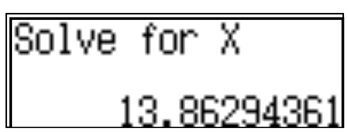
The figure below shows the graphical solution of $2 = (1.05)^x$.



b. This time, we solve the equation $2P = Pe^{0.05x}$. This implies $2 = e^{0.05x}$ or equivalently, $e^{0.05x} - 2 = 0$. We use the following calculator commands:

[Operations]

- Press Mode. Select 1: COMP — [MODE] [1]
- Enter $e^{0.05x}$ — [ALPHA] [x10^x] (e) [x■] [•] [0] [5] [ALPHA] [)] (X) [▶]
- Enter -2 — [−] [2]
- Enter = 0 — [ALPHA] [CALC] (=) [0]
- Press solve button — [SHIFT] [CALC] (SOLVE) [=]



We obtain $x = 13.9$ years.

Using the tabular feature of the calculator, we can verify this result by generating the future values of the money as follows:

[Operations]

- Press Mode. Select 7: TABLE — [MODE] [7]

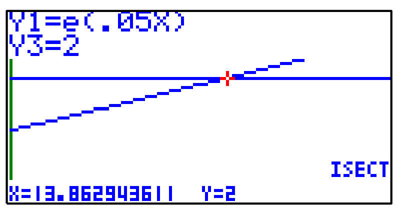
  Enter the given function $e^{0.05x}$

- Enter $e^{0.05x}$ — [ALPHA] [$\times10^x$] (e) [$x^{\blacksquare}$] [•] [0] [5] [ALPHA] [)] (X) [▶] [=]
- Enter start value, 13 — [1] [3] [=]
- Enter end value, 14 — [1] [4] [=]
- Enter step value, 0.1 — [•] [1] [=]

We obtain the following table:

Observe that when $x = 13.9$, money is about 2.004 or approximately 2. It will take approximately 13.9 years for money to double if it is invested at 5% interest compounded continuously.

The figure below illustrates the graphical solution of $2 = e^{0.05x}$.



EXERCISES

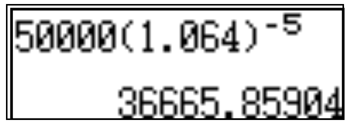
1. A note will pay $50,000 at maturity 5 years from now. How much should you be willing to pay for the note now if money is worth 6.4%
a. compounded annually?
b. compounded continuously?

Solution:

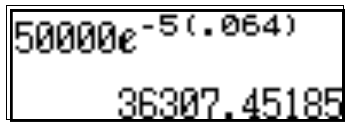
a. We determine the answer using the following formula: present value = $50{,}000(1.064)^{-5}$. We obtain the answer $36,665.86.

[Operations]

- Press Mode. Select 1: COMP — [MODE] [1]
- Enter 50,000 — [5] [0] [0] [0] [0] [•]
- Enter $(1.064)^{-5}$ — [(] [1] [•] [0] [6] [4] [)] [$x^{\blacksquare}$] [(−)] [5] [=]



b.This time, we use the formula: present value = $50{,}000e^{-5(.064)}$. The answer is $36,307.45.



[Operations]

- Press Mode. Select 1: COMP — [MODE] [1]
- Enter 50,000 — [5] [0] [0] [0] [0] [•]
- Enter $e^{-5(.064)}$ — [ALPHA] [×10^x] (e) [x^■] [(−)] [5] [(] [•] [0] [6] [4] [)] [=]

2. A man with $20,000 to invest decides to diversify his investments by placing $10,000 in an account that earns 7.2% compounded continuously and $10,000 in an account that earns 8.4% compounded annually. Determine how long will it take for his total investment in the two accounts to grow to $35,000.

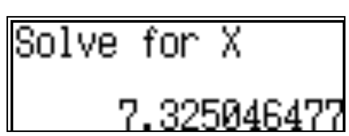
Solution:

Let x be the number of years it will take for his total investment in the two accounts to grow to $35,000. We solve the equation $10{,}000e^{(.072)x} + 10{,}000(1.084)^{x} = 35{,}000$ or equivalently, $10{,}000e^{(.072)x} + 10{,}000(1.084)^{x} - 35{,}000 = 0$.

We use the following calculator commands:

- Press Mode. Select 1: COMP — [MODE] [1]
- Enter $10{,}000e^{0.072x}$ — [1] [0] [0] [0] [0] [0] [ALPHA] [×10^x] (e) [x^■]
- Continue — [•] [0] [7] [2] [ALPHA] [)] (X) [▶]
- Enter $+10{,}000(1.084)^{x}$ — [+] [1] [0] [0] [0] [0] [0] [(]
- Continue — [1] [•] [0] [8] [4] [)] [x^■] [ALPHA] [)] (X) [▶]
- Enter $-35{,}000$ — [−] [3] [5] [0] [0] [0]
- Enter = 0 — [ALPHA] [CALC] (=) [0]
- Press solve button — [SHIFT] [CALC] (SOLVE) [=]

The answer is approximately 7.32 years.



Appendix

Calculating with the FX-991 ES Plus

Turning Power On and Off. Press [ON] to turn the calculator *on*, and [SHIFT] [AC] to turn it *off*.

Second Function Keys. The features printed in yellow are accessed by pressing [SHIFT] then the desired key.

Alpha Keys. The items in red are accessed by pressing [ALPHA] then the desired key.

Selecting the Mode. To select the mode, press [MODE]. The screen shows a menu. Each choice leads to different calculator capabilities. You can select the mode by inputting the number marked on the left.

```
1:COMP   2:CMPLX
3:STAT   4:BASE-N
5:EQN    6:MATRIX
7:TABLE  8:VECTOR
```

Set Up Menu. Press [SHIFT] [MODE] (SETUP). A menu is displayed on the screen. A choice given is accessed by inputting the number on the left. Use [▼] [▲] to navigate between the two page menu.

```
               ▼
1:MthIO 2:LineIO
3:Deg   4:Rad
5:Gra   6:Fix
7:Sci   8:Norm
```

```
               ▲
1:ab/c  2:d/c
3:CMPLX 4:STAT
5:Disp  6:◀CONT▶
```

Natural Display. Use [SHIFT] [MODE] [1] (MthIO). Choosing Natural Display makes it possible to input and display fractions, irrational numbers and particular functions just as they appear in a book. For instance,

$$\frac{\log_7(49)}{3}$$

.

Linear Display. Press [SHIFT] [MODE] [2] (LineIO).The calculation results are displayed in linear format. Fractions and other expressions are displayed in a single line. For example,

```
1⌟3+13⌟15
                6⌟5
```

Degree, Radian or Gradian. Use [SHIFT] [MODE] [3] (Deg) , [SHIFT] [MODE] [4] (Rad) or [SHIFT] [MODE] [5] (Gra), respectively. These commands are used to specify the degrees, radians or gradians as the angle unit for value input and calculation result display.

Fix, Scientific or Norm. We use SHIFT MODE 6 (Fix) , SHIFT MODE 7 (Sci) or SHIFT MODE 8 (Norm). Each command specifies the number of digits for display for calculation results.

ab/c or d/c. Press 1 or 2. This respectively specifies either a mixed fraction or improper fraction for display of fractions in calculations.

CMPLX. Select 3. Choose either 1 or 2. The complex result will be displayed either in rectangular coordinates $(a+bi)$ or polar coordinates $r<\theta$.

STAT. Enter 4. Choose either 1 or 2. The choice specifies whether or not to display a FREQ column in the Stat Mode Stat Editor.

Disp. Enter 5. Select 1 or 2. This specifies whether to display a dot or a comma to represent a decimal point.

CONT. Use 6. This key adjusts the display contrast.

To complete a calculation, we type a command line in the calculator, and then press = key to execute the command line.

Entering Mathematical functions. The variable x is accessed by pressing ALPHA) (X). To enter x^2, we press ALPHA) (X) x^2. For x^3, we have the keys ALPHA) (X) SHIFT x^2 (x^3). On the other hand to enter x^n for some integer n, we use ALPHA) (X) $x^{\blacksquare}$ then value n of the exponent. Here are the other keys to access functions.

Absolute value function
Enter SHIFT hyp (Abs) ALPHA) (X).

Exponential function
b^x ($b > 1$ or $0 < b < 1$): Enter value of b then press $x^{\blacksquare}$ ALPHA) (X); 10^x: enter SHIFT log ($10^{\blacksquare}$); e^x : enter SHIFT ln ($e^{\blacksquare}$).

Hyperbolic function
Press hyp. Choose 1 (sinh), 2 (cosh), 3 (tanh), 4 ($\sinh^{-1}$), 5 ($\cosh^{-1}$) and 6 ($\tanh^{-1}$).

Logarithmic function
$log_a\, x$: Enter ALPHA) (X) then specify the value of base a; $\ln x$: enter ln ALPHA) (X).

Trigonometric function
Press either sin, cos, tan, SHIFT sin ($\sin^{-1}$), SHIFT cos ($\cos^{-1}$) or SHIFT tan ($\tan^{-1}$) followed by ALPHA) (X) .

COMP Mode is for the purpose of general calculations.

To select COMP Mode press [MODE] [1].

Fractions/Decimals. Use [▭/▭] or [SHIFT] [▭/▭] (■▭/▭) for fractions and quotients. To convert decimals to fractions use [S⇔D]. To change from mixed number to improper fraction use [SHIFT] [S⇔D] ($a\frac{b}{c} \Leftrightarrow \frac{d}{c}$).

Percent. Use [SHIFT] [(] **(%)** in dealing with percentage. For example, to determine 20% of 500, we enter 500 x 20 [SHIFT] [(] **(%).**

Factorial Notation. Use [SHIFT] [x^{-1}] **(!)**

Summation. (Natural Display Syntax). Use [SHIFT] [log▪▯] (Σ▪), enter the function and lower/upper limit of the summation.

Polar/Rectangular Coordinates. Pol converts rectangular coordinates to polar coordinates, whil Rec converts polar coordinates to rectangular coordinates. The angle unit is specified before performing the calculations. In getting the polar coordinates we enter [SHIFT] [+] (Pol) then value of x, y. In obtaining the rectangular coordinates we use [SHIFT] [−] (Rect) then value of r, θ.

Permutation/Combination. Use [SHIFT] [×] (n*P*r) and [SHIFT] [÷] (nCr) keys.

Solve Command. The calculator uses Newton's Law to approximate the solution of equations. To solve for x in a given equation, we use the following syntax: equation, variable then [SHIFT] [CALC] (SOLVE).

For example, we have

[ALPHA] [)] (X) [x^2] [+] [2] [ALPHA] [)] (X) [−] [3] [SHIFT] [)] (,) [ALPHA] [)] (X)

```
X²+2X-3,X
```

[SHIFT] [CALC] (SOLVE)

```
Solve for X
                1
```

Derivative at a Point. (Natural Display Syntax) Press [SHIFT] [∫▭] ($\frac{d}{dx}$▪)

Enter the function and particular x value. For instance, derivative of $f(x) = x^2$ at $x = 2$ is entered as

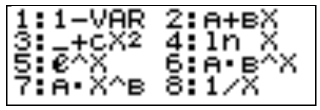
```
d/dx(X²)|x=2
```

.

Definite Integral. (Natural Display Syntax) Press [∫], enter the function and lower/upper limit of the integral.

COMPLEX Number Mode. Press [MODE] [2] to access the complex number mode.

STAT Mode. To begin a statistical calculation, press [MODE] [3]. The following menu is shown:

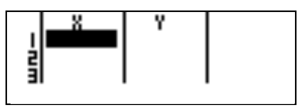


Choose [1] for single variable X calculations, such as mean, standard deviation; [2] for linear regression ($Y = A+BX$), [3] for quadratic regression ($Y = A + BX + CX^2$), [4] for logarithmic equation ($Y = A + B\ln X$), [5] for e exponential regression ($Y = Ae^{BX}$), [6] for (base B) exponential regression ($Y = AB^X$), [7] for power regression ($Y = AX^B$) and [8] for inverse regression ($Y = 1/X$).

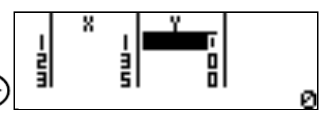
Pressing any of the 8 keys above will access the stat editor. The stat editor is used when inputting data. After inputting the data, access [SHIFT] [1] [5] to get the regression coefficients, or estimated values (press any choice from [1] to [6]). The information will appear where the cursor was previously positioned, after returning [=].

For instance using quadratic regression on the following points (1,1) (3, 8), (5,22) and (7, 50) will require the following keys, namely:

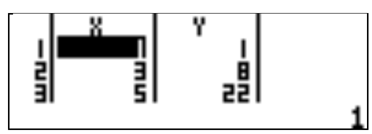
[MODE] [3] [3]



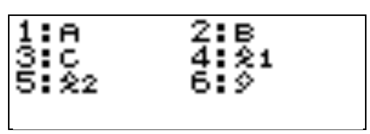
[1] [=] [3] [=] [5] [=] [7] [=] [▲] [▲] [▲] [▲] [▶]



[1] [=] [8] [=] [2] [2] [=] [5] [0] [=]



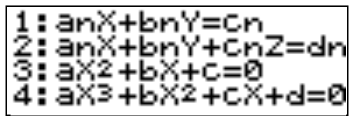
[AC] [SHIFT] [1] [5]



Then the choice is inputted pertaining to the coefficients A,B,C, x or y values.

BASE-N Mode. This accessed by the keys [MODE] [3] and is used for calculations involving specific number systems such as the binary, decimal, hexadecimal and octal number systems.

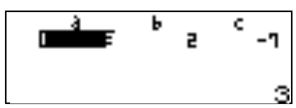
EQUATION Mode. Use [MODE] [5]. A menu giving the type of equation appears on the screen.

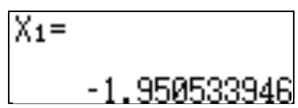


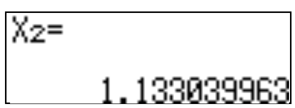
Choose [1] for solving simultaneous linear equations with two unknowns, [2] for solving simultaneous linear equations with three unknowns, [3] for solutions to a quadratic equation and [4] for a cubic equation.

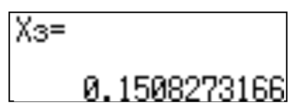
Use the coefficient editor that appears to input coefficient values. After entering all the coefficients, press [=]. This will display a solution. Each time [=] is pressed, another solution is displayed. One can scroll between solutions using [▼] or [▲] keys.

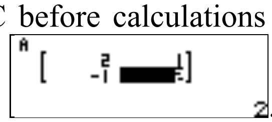
For example, in solving $3x^3+2x^2-7x+1=0$, we input [MODE] [5] [4] then [3] [=] [2] [=] [(-)] [7] [=] [1] [=] [=]. We have the following screen dumps:



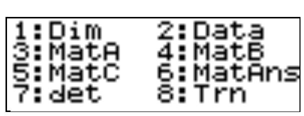






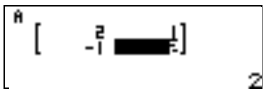
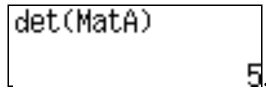
MATRIX Mode. The MATRIX Mode is used for calculations involving 1, 2 or 3 matrices (up to 3 rows by 3 columns). Enter [MODE] [6] to enter the MATRIX Mode. Data is assigned to either Mat A, Mat B or Mat C before calculations are carried out. For instance we press [1] [5] to enter data to 2x2 Mat A . After inputting the data for the matrices included in the calculations, press [AC] to advance to the calculation screen.

The menu pertaining to matrix calculation can be accessed using [SHIFT] [4] (MATRIX).



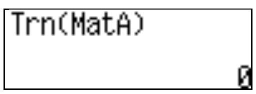
Matrix Calculations:

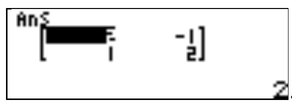
Determinant of a Matrix. Use [SHIFT] [4] (MATRIX) [7] (det) then [SHIFT] [4] to access either (Mat A) [3] , (Mat B) [4] or Mat C [5] .

For example, given  we have 

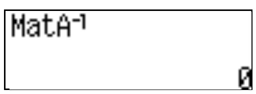
.

Transpose of a Matrix. Use [SHIFT] [4] (MATRIX) [8] (Trn) then [SHIFT] [4] to access either (Mat A) [3] , (Mat B) [4] or Mat C [5].

For instance, 

gives 
.

Inverse of a Matrix. Enter [SHIFT] [4] (MATRIX) to access (Mat A) [3] , (Mat B) [4] or Mat C [5] then [x^{-1}].

For example, 

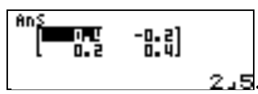
will yield 
. .

TABLE Mode. Use [MODE] [7] to access the TABLE Mode.Input the function $f(x)$ then press [=]. For the Start? prompt, input the lower limit of x (default value is 1); the End? Prompt, enter the upper limit of x (default value is 5) and for the Step? Prompt input the increment step (default is 1).

VECTOR Mode. This mode is accessed by [MODE] [8]. This is used to perform 2-dimensional and 3-dimensional vector calculations.

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References

Barnett, Raymond, et al. Calculus for Business, Economics, Life Sciences and Social Sciences 12th Edition. Pearson Education, Inc, Singapore (2010).

Barry, M.D.J and Mustoe, L.R. Mathematics in Engineering and Science. Wiley and Sons, Ltd, USA (1998).

Beecher, Judith, et al. Algebra and Trigonometry 2nd Edition. Addison Wesley Longman, Inc, USA (2001).

Bittinger, Marvin et al. Algebra and Trigonometry, Graphs and Models 2nd Edition. Addison Wesley Longman, Inc, USA (2001).

De Las Penas, M.L.A.N., Fong, M.C., Harradine, A. , Kissane, B., Preiss, R. and Yang, W.C. Casio Scientific Calculator ES Series Activities, Casio Computer Co. Ltd, Japan (2005).

De Las Penas, M.L.A.N. Fong, M.C., Jozef, H. and Yang, W.C. Casio Graphic Calculator FX-9860 Series Activities, Casio Computer Co. Ltd, Japan (2005).

Hoffmann, Laurence, et al. Calculus for Business, Economics, Life Sciences and Social Sciences 8th Edition. Mc Graw Hill, USA (2004).

Johnson, Richard and Bhattacharyya, Gouri. Statistics, Principles and Methods. Wiley and Sons, USA (1996).

Lial, Margaret et al. Finite Mathematics 7th Edition. Pearson Education Inc, USA (2002).

Mann, Prem. Introductory Statistics 5th Edition, Wiley and Sons, Inc, USA (2004).

Mendenhall, William et al. Introduction to Probability and Statistics 11th Edition. Brooks and Cole, USA (2003).

Miller, Charles, et al. Mathematical Ideas 9th Edition. Addison Wesley Educational Publishers, USA (2001).

Stewart, James. Calculus: Early Transcendentals 7th Edition. Brooks and Cole, USA (2009).

Stewart, James. College Algebra, 4th Edition. Brooks and Cole, USA (2004).

Swokoski, E. and Cole, J. Algebra and Trigonometry with Analytic Geometry 10th Edition. Wadsworth Group, USA (2002).

Waner, Stefan and Costenoble, Steven.Finite Mathematics 3rd Edition. Brooks and Cole, USA (2004).

Waner, Stefan and Costenoble, Steven. Finite Mathematics and Applied Calculus 2nd Edition. Wadsworth Group, Brooks and Cole, USA (2001).